Sample Test 3 Solutions!

1. Find the value of the triple integral

$$\iiint_T f(x,y,z)\,dV,$$

where $f(x,y,z) = x^2$ and T is the tetrahedron bounded by the coordinate planes and the first octant part of the plane with equation $x + y + z = 1$.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2\,dz\,dy\,dx = \frac{1}{60}$$

2. Compute the mass and centroid of the plane lamina bounded by $y = 0$ and $y = \sin x$ for $0 \le x \le \pi$ given that it has density given by the function $\delta(x,y) = x$.

$$m = \int_0^\pi \int_0^{\sin x} x\,dy\,dx = \int_0^\pi x \sin x\,dx = (-x\cos x + \sin x)\big|_0^\pi = \pi.$$

$$\overline{x} = \frac{1}{\pi}\int_0^\pi \int_0^{\sin x} x^2\,dy\,dx = \pi - \frac{4}{\pi}$$

$$\overline{y} = \frac{1}{\pi}\int_0^\pi \int_0^{\sin x} xy\,dy\,dx = \frac{\pi}{8}$$

Thus the mass is π and the centroid is $\left(\pi - \frac{4}{\pi}, \frac{\pi}{8}\right)$.

3. Find the volume of the solid under the surface

$$z = \frac{xy}{1 + x^2y^2}$$

and over the region bounded by $xy = 1, xy = 5, x = 1$, and $x = 5$.

Let $u = xy$ and $v = x$. Then $x(u,v) = v$ and $y(u,v) = \frac{u}{v}$. The Jacobian of the transformation is:

$$\left\| \begin{array}{cc} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{array} \right\| = \frac{1}{v}.$$

Then the volume is:

$$\begin{aligned}\int_1^5 \int_1^5 \frac{u}{1+u^2}\frac{1}{v}\,du\,dv &= \left(\int_1^5 \frac{1}{v}\,dv\right)\left(\int_1^5 \frac{u}{1+u^2}\,du\right)\\ &= (\ln 5)\left(\frac{1}{2}\ln\left(1+u^2\right)\right)\Big|_1^5\\ &= \frac{1}{2}(\ln 5)(\ln 26 - \ln 2) = \ln\sqrt{65}.\end{aligned}$$

4. Compute the integral

$$\iiint_R z(x^2+y^2)^{(-1/2)}\,dx\,dy\,dz,$$

where R is the region bounded above by the plane $z = 2$ and below by the surface $2z = x^2 + y^2$.

$$\int_0^{2\pi}\int_0^2\int_{r^2/2}^2 \frac{z}{r} r\,dz\,dr\,d\theta = \frac{32\pi}{5}.$$

5. Find the amount of work done moving in a straight line from $(1,0,2)$ to $(3,4,1)$ in the force field $\mathbf{F}(x,y,z) = \langle 2xy, x^2+2, y\rangle$.

Parametrize C by $\mathbf{r}(t) = \langle 1+2t, 4t, 2-t\rangle$, $0 \le t \le 1$. Then $\mathbf{F}(t) = \langle 8t+16t^2, 3+4t+4t^2, 4t\rangle$ and $\mathbf{r}'(t) = \langle 2, 4, -1\rangle$. We then set up the integral:

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\mathbf{r}'\,dt = \int_0^1 \left(12+38t+48t^2\right)\,dt = 42.$$

6. Use the fundamental theorem of line integrals to compute

$$\int_C \langle y, x\rangle \cdot \mathbf{T}\,ds,$$

where C is any path from $(0,0)$ to $(2,4)$.

$\nabla(xy) = \langle y, x\rangle$, so $f(x,y) = xy$ is a potential function for the vector field. By the fundamental theorem,

$$\int_C \langle y, x\rangle \cdot \mathbf{T}\,ds = f(2,4) - f(0,0) = 8.$$

7. Use Green's Theorem to find the work done by the force field $\mathbf{F}(x, y) = \langle 3y - 4x, 4x - y \rangle$ when an object moves once counterclockwise around the ellipse $4x^2 + y^2 = 4$.

$$W = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\frac{\partial}{\partial x}(4x - y) - \frac{\partial}{\partial y}(3y - 4x) \right) dA = 2\pi.$$

Challenge! Let $\mathbf{F}(x, y) = \langle x^2 y, xy^3 \rangle$. Minimize the amount of work done to move from $A(0, 0)$ to $B(1, 1)$ and find the path that minimizes this work. Hint: The path from A to B can be expressed in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, where $x(t)$ and $y(t)$ are both power series. What restrictions are there on the variables?