Solutions to Sample Final

1. True

2. False: X and Y could be perpendicular.

3. Use the second parameterization: note where the curve is at $t = 0$ and $t = \pi/2$.

$$\begin{aligned}
\oint_C -y dx + x dy &= \int_0^{2\pi} ((-4\sin t)(-4\sin t) + (4\cos t)(4\cos t))dt \\
&= \int_0^{2\pi} (16\sin^2 t + 16\cos^2 t)dt \\
&= \int_0^{2\pi} 16 dt = 32\pi
\end{aligned}$$

Thus the net flow of $\{-y, x\}$ is counterclockwise and the net flow of $\{x, y\}$ is outward.

4. Not all path integrals are path independent. This is only guaranteed if the field is a gradient field.

5. If the region is easiest to describe setting it up as a double integral (e.g. a rectangular region), then a path integral is best calculated as a double integral. However, if you are given a double integral with a region that's not as nice to integrate over (e.g. an ellipse), then it is best to compute the path integral over the boundary of the region.

6. Against: Does $X \cdot Y$ form an acute or obtuse angle?

7. The point $(x_0, y_0, f(x_0, y_0))$ is a local maximum, so gradient vectors in its vicinity will be pointing toward it, so the net flow of the gradient field across C is inward.

8. The field has no singularities, so we can find the flow across the curve

using the divField.

$$\begin{aligned} \text{divField}(x, y) &= \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} \\ &= 3x^2 - 1 \end{aligned}$$

So the net flow is

$$\int_0^1 \int_{-2}^1 (3x^2 - 1) dx dy = \int_0^1 (x^3 - x)|_{-2}^1 dy = \int_0^1 (0 - (-6)) dy = 6.$$

The net flow is therefore outward.

9. $\bigtriangledown f(1, 2, 0) = \{ye^z, xe^z, xye^z\}(1, 2, 0) = \{2, 1, 2\}$

10. Since the gradient points in the direction of greatest increase, you can make a trajectory from any point on the surface whose tangent vectors are the gradient vectors, and the trajectory will eventually rerach the highest crest. Similarly, by taking a trajectory from any point such that the tangent vectors are the opposite of the gradient, you will eventually reach the deepest dip.

11. i) flow across
ii) flow along
iii) flow along
iv) flow along
v) flow across
vi) flow along
vii) flow across

12. The area conversion is $dxdy = rdrdt$. So the double integral is:

$$\begin{aligned} \int \int_R (x^2 + y^2) dx dy &= \int_0^{2\pi} \int_0^2 (r^2 \cos^t + r^2 \sin^2 t) r dr dt \\ &= \int_0^{2\pi} \int_0^2 r^3 dr dt \\ &= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 dt \\ &= \int_0^{2\pi} 4 dt = 8\pi. \end{aligned}$$

13. Take $u = x + y$ and $v = x - y$. Then the bounds of R on $uv-$paper are $0 \leq u \leq 2$ and $1 \leq v \leq 2$. Solving x and y in terms of u and v can be obtained by adding the two equations.

$$u + v = 2x \text{ so } x = \frac{u+v}{2}$$

$$u - v = 2y \text{ so } y = \frac{u-v}{2}$$

(Why the heck did Francona have a bench warmer pinch hit for Nixon?!)
Verify that the area conversion factor is $\frac{1}{2}$. The integral is then
(The pinch hitter struck out. Figures.)

$$\begin{aligned}
\int_1^2 \int_0^2 \frac{u+v}{2}\frac{1}{2} du dv &= \int_1^2 \left(\frac{u^2}{8} + \frac{uv}{4} \right) \Big|_0^2 dv \\
&= \int_1^2 \left(\frac{1}{2} + \frac{v}{2} \right) dv \\
&= \left(\frac{v}{2} + \frac{v^2}{4} \right) \Big|_1^2 \\
&= (1+1) - (1/2 + 1/4) = \frac{5}{4}
\end{aligned}$$

14. Use the substitution $x = s\cos t$, $y = 2s\sin t$. Then $0 \leq s \leq 1$ and $0 \leq t \leq 2\pi$. The area conversion factor is $2s$. So substituting everything in we get:

$$\begin{aligned}
\int\int_R e^{-(x^2+(y/2)^2)} dx dy &= \int_0^{2\pi} \int_0^1 2se^{-s^2} ds dt \\
&= \int_0^{2\pi} -e^{-s^2} \Big|_0^1 dt \\
&= \int_0^{2\pi} \left(1 - e^{-1}\right) dt \\
&= 2\left(1 - e^{-1}\right)\pi
\end{aligned}$$

15. a) $\int_0^1 \int_0^{\sqrt{x}} f(x) dy dx$

$\int_0^1 \int_{y^2}^1 f(x)dydx$
b) This measures the area of the region, or the volume of the "cylinder" with height 1 whose base is the region.
c) Since $x^2 + y^3$ is positive over the whole region, this integral measures the volume under the surface $z = x^2 + y^3$ and over the region R.
(Crap, the red Sox lost.)

16.

$$\int_0^4 \int_0^{6-3z/2} \int_0^{6-y-3z/2} xdxdydz$$

17. $r^* = 5$, $0 \le s \le \pi$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

18. $0 \le t \le 2\pi$. For s, we know that $s_{high} < \pi/2$, so make a good guess. I'd say $0 \le s \le \frac{\pi}{4}$, but any reasonable answer will do since we don't have an exact value for the lowest point of the surface.

19. We didn't cover spherical coordinates in this class, so I won't ask a question like this on the exam. But if you're curious, r is the radius (considered three-dimensionally), by dropping a perpendicular from the point to the z-axis and working from $z = r\cos s$, s is the angle formed by the z-axis and the radius. Lastly, by dropping a perpendicular from the point to the xy-plane, projecting the radius as well, then t is the same as the angle t (or θ) is polar coordinates.

20. The volume conversion factor is $r^2 \sin s$, which you can check by hand

from the substitutions in (19). Then we have

$$\int\int\int_R (x^2+y^2+z^2)dxdydz = \int_0^{\pi}\int_0^{2\pi}\int_0^4 r^2(r^2\sin s)drdtds$$
$$= \int_0^{\pi}\int_0^{2\pi} \frac{r^5}{5}\sin s\Big|_0^4 dtds$$
$$= \int_0^{\pi}\int_0^{2\pi} \frac{1024}{5}\sin sdtds$$
$$= \int_0^{\pi} \frac{2048\pi}{5}\sin sds$$
$$= -\frac{2048\pi}{5}\cos s\Big|_0^{\pi}$$
$$= \frac{4096\pi}{5}.$$