Name (please print):

Instructions: Show all work. No calculators, books, notes, etc.

1. (a) What does it mean for a vector field $\vec{F}(x, y) = \{m(x, y), n(x, y)\}$ to be a gradient field? What is one property of gradient fields. [8 pts.]

   Solution: A vector field is a gradient field if there is a function $f(x, y)$ such that $\nabla f = \vec{F}$, that is, if $\frac{\partial f}{\partial x} = m$ and $\frac{\partial f}{\partial y} = n$.

   Some properties of gradient fields are that they are irrotaional (except possibly around singularities), path integrals in these fields are independent of path, and they point in the direction of greatest increase of $f(x, y)$.

   (b) Is $\vec{F}(x, y) = \{3x^2 + 2y, 2x + 6y\}$ a gradient field? If so, find a function $f(x, y)$ such that $\nabla f = \vec{F}$. [7 pts.]

   Solution: Check to see if $\frac{\partial(3x^2+2y)}{\partial y} = \frac{\partial(2x+6y)}{\partial x}$. $2 = 2$. Check. Now let's find the function $f(x, y)$ such that $\nabla f(x, y) = \{3x^2 + 2y, 2x + 6y\}$.

   Using the definition of the gradient (first half of part (a)):

$$
\begin{aligned}
f(x, y) &= \int (3x^2 + 2y)\,dx & &= \int (2x + 6y)\,dy \\
&= x^3 + 2xy + C_y & &= x^2 + 3y^2 + C_x \\
&= x^3 + 2xy + 3y^2 & & .
\end{aligned}
$$

2. Calculate the flow of the field $\{-y, x\}$ along the ellipse $\frac{x^2}{9} + y^2 = 1$ using a single integral. Is the flow clockwise or counterclockwise? [10 pts.]

   Solution: The general flow along formula is $\oint_C \vec{F}(x, y) \cdot \vec{T}\,dt$, where $\vec{F}(x, y)$ is the field and $\vec{T}$ is the tangent of the curve C. This integral measures how much the field goes with the tangent, i.e. along the curve. We'll apply this formula to this example.

Set $x(t) = 3\cos(t)$ and $y(t) = \sin(t)$, then:

$$\begin{aligned}\oint_C \vec{F}(x,y)\cdot\vec{T}\,dt &= \int_0^{2\pi} \{-\sin t, 3\cos t\}\cdot\{x'(t), y'(t)\}\,dt\\ &= \int_0^{2\pi} \{-\sin t, 3\cos t\}\cdot\{-3\sin t, \cos t\}\,dt\\ &= \int_0^{2\pi} \left(3\sin^2 t + 3\cos^2 t\right)\,dt\\ &= \int_0^{2\pi} 3\,dt = 6\pi\ .\end{aligned}$$

So the flow is counterclockwise.

3. Set up, but do not evaluate, both integrals to compute the volume of the solid under the surface $z = x^2 + y^2$ and over the triangle with corners at $(0,0)$, $(0,1)$, and $(1,1)$. [10 pts.]

   Solution: Draw the triangle, which is bounded by $x = 0$, $y = 1$, and $y = x$, then the integrals are:

$$\int_0^1\int_0^y (x^2+y^2)\,dx\,dy$$

$$\int_0^1\int_y^1 (x^2+y^2)\,dy\,dx$$

4. The Gauss-Green Theorem states that

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)\,dx\,dy = \oint_C (m(x(t),y(t))x'(t) + n(x(t),y(t))y'(t))\,dt\ .$$

   Use this theorem to set up (but don't evaluate) a single integral to compute the volume of the solid bounded by the xy-plane and the surface $z = 2 - x^2 - y^2$. [15 pts.]

   Solution: The integral to compute the volume is $\iint_R (2 - x^2 - y^2)\,dx\,dy$, where R is the region in the xy-plane that the surface sits. The intersection of the surface and the plane is $0 = 2 - x^2 - y^2$, so R is the region bounded by $x^2 + y^2 = 2$. Thus C is parameterized $x(t) = \sqrt{2}\cos t$, $y(t) = \sqrt{2}\sin t$, $0 \leq t \leq 2\pi$.

Line up the double integrals:

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dx\, dy = \iint_R \left(2 - x^2 - y^2\right) dx\, dy \ ,$$

so that

$$2 - x^2 - y^2 = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \ .$$

We make choices $m(x, y)$ and $n(x, y)$ that make this work. Let $m(x, y) = 0$, so that $2 - x^2 - y^2 = \frac{\partial n}{\partial x}$, and $n(x, y) = 2x - \frac{x^3}{3} - xy^2$.

Now substitute everything into the single integral and we get:

$$\int_0^{2\pi} \left(2(\sqrt{2}\cos t) - \frac{(\sqrt{2}\cos t)^3}{3} - (\sqrt{2}\cos t)(\sqrt{2}\sin t)^2\right)\left(\sqrt{2}\cos t\right) dt \ .$$

5. Let $\vec{F}(x, y) = \{\sin(x) - y, \cos(y) - e^{-x}\}$. Find $\text{rot}\vec{F}(x, y)$. What does this result tell you about the flow of $\vec{F}(x, y)$ around any closed curve? [13 pts.]

   Solution:

   $$\text{rot}\vec{F} = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = -(-e^{-x}) - (-1) = 1 + e^{-x} \ .$$

   Since $\text{rot}\vec{F} > 0$ for all x, the flow of $\vec{F}$ along any closed curve is counterclockwise.

6. The field $\vec{F}(x, y) = \{-x^2y + 2xy, 4x^3y + xy^2\}$ models the flow of a fluid, possibly with sinks and sources. Find the net flow of this fluid across the square with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. [12 pts.]

   Solution:

   $$\text{div}\vec{E} = \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} = -2xy + 2y + 4x^3 + 2xy = 2y + 4x^3 \ .$$

So the flow of $\vec{F}$ around the square is

$$\begin{aligned}\int_0^1 \int_0^1 (2y + 4x^3)\, dx\, dy &= \int_0^1 2xy + x^4\Big|_0^1 \, dy \\ &= \int_0^1 (2y+1)\, dy \\ &= y^2 + y\Big|_0^1 = 1 + 1 = 2 \ .\end{aligned}$$

7. Set up, but do not evaluate, the integral to find the volume of the solid under $z = \frac{xy}{1+x^2y^2}$ and over the region bounded by $xy = 1$, $xy = 4$, $x = 1$, and $x = 2$ using an appropriate change of variables. [13 pts.]

Solution: Let $u = xy$ and $v = x$, $1 \le u \le 4$, $1 \le v \le 2$. Then $x = v$ and $y = \frac{u}{x} = \frac{u}{v}$. Now for our area conversion (a.k.a. "fudge") factor:

$$A_{x,y}(u,v) = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\| = \left\| \begin{array}{cc} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{array} \right\| = \left| -\frac{1}{v} \right| = \frac{1}{v} \ .$$

Since $v > 0$, we can leave off the absolute value bars. Let's set up the integral. (Notice that $x^2y^2 = (xy)^2 = u^2$.)

$$\int_1^2 \int_1^4 \left(\frac{u}{1+u^2} \right) \left(\frac{1}{v} \right) du\, dv \ .$$

8. Set up, but do not evaluate, the integral to compute the volume of the solid under $z = x^2 + 4y^2$ and over the region bounded by $\frac{x^2}{16} + \frac{y^2}{4} = 1$ using an appropriate change of variables. [12 pts.]

Solution: Change the coordinate system to $x(r,t) = 4r\cos t$, $y(r,t) = 2r\sin t$, $0 \le r \le 1$, $0 \le t \le 2\pi$. Now for our area conversion (a.k.a. "fudge") factor:

$$A_{x,y}(u,v) = \left\| \begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{array} \right\| = \left\| \begin{array}{cc} 4\cos t & -4r\sin t \\ 2\sin t & 2r\cos t \end{array} \right\| = \left| 8r\cos^2 t + 8r\sin^2 t \right| = 8r \ .$$

So the integral is:

$$\int_0^{2\pi} \int_0^1 \left((4r\cos t)^2 + 4(2r\sin t)^2 \right) (8r)\, dr\, dt = \int_0^{2\pi} \int_0^1 128r^3\, dr\, dt \ .$$