1. Is $\{2x+y, x+2y\}$ a gradient field? If so, find the function $f(x,y)$ such that $\nabla f(x,y) = \{2x+y, x+2y\}$. [6 pts.]

   Solution: Check to see if $\frac{\partial(2x+y)}{\partial y} = \frac{\partial(x+2y)}{\partial x}$. $1 = 1$. Check. Now let's find the function $f(x,y)$ such that $\nabla f(x,y) = \{2x+y, x+2y\}$.

   Using the definition of the gradient:

$$\begin{aligned} f(x,y) &= \int (2x+y)\,dx & &= \int (x+2y)\,dy \\ &= x^2 + xy + C_y & &= xy + y^2 + C_x \\ &= x^2 + xy + y^2 & &. \end{aligned}$$

2. Calculate the flow of the field $\{x, y\}$ across the circle $x^2 + y^2 = 16$ using a single integral. Is the flow inwards or outwards? [8 points]

   Solution: The general flow across formula is $\oint_C \vec{F}(x,y) \cdot \vec{n}\,dt$, where $\vec{F}(x,y)$ is the field and $\vec{n}$ is the outward unit normal of the curve C. This integral measures how much the field goes with the outward normal, i.e. across the curve. We'll apply this formula to this example.

   Set $x(t) = 4\cos(t)$ and $y(t) = 4\sin(t)$, then:

$$\begin{aligned} \oint_C \vec{F}(x,y) \cdot \vec{n}\,dt &= \int_0^{2\pi} \{4\cos t, 4\sin t\} \cdot \{y'(t), -x'(t)\}\,dt \\ &= \int_0^{2\pi} \{4\cos t, 4\sin t\} \cdot \{4\cos t, 4\sin t\}\,dt \\ &= \int_0^{2\pi} \left(16\cos^2 t + 16\sin^2 t\right)\,dt \\ &= \int_0^{2\pi} 16\,dt = 32\pi \ . \end{aligned}$$

   So the flow is outwards.

3. Let C_1 and C_2 be paths from $(0,0)$ to $(1,1)$ defined by $\vec{r_1}(t) = \{t,t\}$ and $\vec{r_2}(t) = \{t,t^2\}$, respectively, where $0 \leq t \leq 1$. Why does [10 pts.]

$$\int_{C_1} y\sin(xy)\,dx + x\sin(xy)\,dy = \int_{C_2} y\sin(xy)\,dx + x\sin(xy)\,dy \ ?$$

Solution: These integrals can be rewritten:

$$\int_{C_1} \{y\sin(xy), x\sin(xy)\} \cdot \{dx, dy\}$$

and

$$\int_{C_2} \{y\sin(xy), x\sin(xy)\} \cdot \{dx, dy\} \ ,$$

so both path integrals are being computed over the field $\{y\sin(xy), x\sin(xy)\}$ from $(0,0)$ to $(1,1)$. Notice that $\{y\sin(xy), x\sin(xy)\} = \nabla(\sin(xy))$, so that the field the paths are in is a gradient field. In a gradient field, we are guaranteed path-independence of path integrals, so these two path integrals are equal as a result.

4. Find the volume of the prism under $z = 6y$ and over the region bounded by $y = x^2$ and $y = 2 - x^2$. [8 pts.]

Solution: First we find the bounds for x in the region by finding the intersection points of the two parabolas:

$$x^2 = 2 - x^2 \Rightarrow 2x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \ .$$

Now we set up and compute the integral:

$$\begin{aligned}
\int_{-1}^{1}\int_{x^2}^{2-x^2} 6y\,dy\,dx &= \int_{-1}^{1} 3y^2\Big|_{x^2}^{2-x^2}\,dx \\
&= \int_{-1}^{1} \left(3(2-x^2)^2 - 3(x^2)^2\right)\,dx \\
&= \int_{-1}^{1} \left(12 - 12x^2 + 3x^4 - 3x^4\right)\,dx \\
&= \int_{-1}^{1} \left(12 - 12x^2\right)\,dx \\
&= 12x - 4x^3\Big|_{-1}^{1} \\
&= (12-4) - (-12+4) = 16 \ .
\end{aligned}$$

5. The Gauss-Green Theorem states that

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dx\, dy = \oint_C (m(x(t), y(t))x'(t) + n(x(t), y(t))y'(t))\, dt \ .$$

Set up, but do not evaluate, the integral to compute $\iint_R (2x + 3y^2)\ dx\, dy$, where R is the region bounded by the ellipse $x^2 + \frac{y^2}{4} = 1$. [10 pts.]

Solution: Line up the double integrals:

$$\iint_R \left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right) dx\, dy = \iint_R \left(2x + 3y^2\right)\ dx\, dy,$$

so that

$$2x + 3y^2 = \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}.$$

We make choices $m(x, y)$ and $n(x, y)$ that make this work. There are a few ways we can choose these two functions, and I'll make note of a couple.

Method 1: Set $2x = \frac{\partial n}{\partial x}$ and $3y^2 = -\frac{\partial m}{\partial y}$. Then notice that $n(x, y) = x^2$ and $m(x, y) = -y^3$.

Method 2: This is the way we usually did this. Set $m(x, y) = 0$, then $2x + 3y^2 = \frac{\partial n}{\partial x}$, so that $n(x, y) = x^2 + 3xy^2$.

Parameterize:
Now we parameterize C, the boundary of R: $\{x(t), y(t)\} = \{\cos t, 2\sin t\}$. lastly, we set up the integral using the Gauss-Green Theorem as given.

Method 1:

$$\int_0^{2\pi} \left(-(2\sin t)^3(-\sin t) + \cos^2 t(2\cos t)\right)\ dt = \int_0^{2\pi} \left(8\sin^4 t + 2\cos^3 t\right)\ dt$$

Method 2:

$$\int_0^{2\pi} \left((\cos^2 t + 3(\cos t)(2\sin t)^2)(2\cos t\right)\ dt = \int_0^{2\pi} \left(2\cos^3 t + 24\cos^2 t\sin^2 t\right)\ dt$$

6. Evaluate $\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy$ by reversing the order of integration. [8 pts.]

Solution: This is one of those examples in which I noted that is impossible to integrate directly, so switching the order of integration is the only way to compute this by hand. (OK, if you want to compute the Taylor series of $\frac{\sin x}{x}$, and do a term-by-term integration, then it's possible that way, and probably wouldn't be too bad. I'll have to try that for fun.) The easiest way to start this is to sketch a picture of the region of integration. Then after reversing the order of integration we have:

$$\begin{aligned}\int_0^\pi \int_0^x \frac{\sin x}{x}\, dy\, dx &= \int_0^\pi \frac{\sin x}{x} y \bigg|_0^x dx \\ &= \int_0^\pi \sin x\, dx = -\cos x|_0^\pi \\ &= -\cos\pi - (-\cos 0) = 1 + 1 = 2.\end{aligned}$$

Note: If you do the whole Taylor sum approach, you end up with the following identity:

$$2 = \sum_{j=0}^{\infty} \frac{(-1)^j \pi^{2j+2}}{(2j+2)!} \quad .$$

That's pretty cool.

7. If $\vec{F}(x,y) = \{-e^x\cos(y), e^x\sin(y)\}$, find $\text{rot}\vec{F}(x,y)$. What does this result tell you about the flow of $\vec{F}(x,y)$ along the circle $x^2+y^2=1$? [10 pts.]

Solution:

$$\begin{aligned}\text{rot}\vec{F} &= \frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} \\ &= e^x \sin y - (-(-e^x \sin y)) = 0 \quad .\end{aligned}$$

Since $\text{rot}\vec{F} = 0$, this field is irrotational, except for singularities, but there are no singularities, so the flow of $\vec{F}$ along the circle (or any closed curve for that matter) is 0.

8. The field $\vec{E}(x,y) = \left\{\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right\}$ describes an electric field. Calculate $\text{div}\vec{E}(x,y)$. Where are the sources or sinks (i.e. positive or negative charges) in this field? [14 pts.]

Solution: Use the quotient rule on this problem.

$$\begin{aligned}\text{div}\vec{E} &= \frac{\partial m}{\partial x} + \frac{\partial n}{\partial y} \\ &= \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} + \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} \\ &= \frac{2(x^2+y^2)-2x^2-2y^2}{(x^2+y^2)^2} \\ &= 0 \ .\end{aligned}$$

Since $\text{div}\vec{E} = 0$, no point in the plane can be a source or sink except for possibly at singularities. Notice that $(0,0)$ is the only singularity in this field. By plotting some of the vectors of the field $\vec{E}(x,y)$, we see that there is a source (positive charge) at $(0,0)$. Plot, for example $(0,\pm1)$, $(\pm1,0)$, and $(\pm1,\pm1)$ to get a good idea of what's going on and see that it's a source.

9. Set up, but do not evaluate, the integral to find the flow of the vector field $\{2x^2y + \cos y^2, e^{-x^2} + 4xy^2\}$ across the rhombus bounded by $y = x$, $y = x+4$, $y = -x$, and $y = 4-x$ using an appropriate change of variables. [14 pts.]

Solution: First we'll compute the divergence of the field:

$$\text{div}\{2x^2y + \cos y^2, e^{-x^2} + 4xy^2\} = 4xy + 8xy = 12xy \ .$$

Now we need to change the coordinate system. Notice that by bringing x to the other side of the boundary equations we have nice choices for u and v:
$u(x,y) = y - x$, $0 \le u \le 4$ and $v(x,y) = x + y$, $0 \le v \le 4$.
Solving for x and y, we get:
$x(u,v) = \frac{v-u}{2}$ and $y(u,v) = \frac{u+v}{2}$.
Now for our area conversion (a.k.a. "fudge") factor:

$$A_{x,y}(u,v) = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\| = \left\| \begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right\| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2} \ .$$

Now we can put it all together:

$$\int_0^4 \int_0^4 \frac{1}{2}\left(12\left(\frac{v-u}{2}\right)\left(\frac{u+v}{2}\right)\right) du\, dv.$$

10. Set up, but do not evaluate, the integral to compute the area of the astroid $x^{2/3}+y^{2/3}=4$ using an appropriate change of variables. Simplify the integrand as much as possible. [12 pts.]

Solution: This is the Pittsburg Steelers problem. Graph out three such astroids, arrange them just right, and color them appropriately, and you get their logo. Anyway, the trick with this problem is to get your coordinate changes so that after substituting into the equation $x^{2/3}+y^{2/3}=4$, you end up applying the identity we all know and love: $\cos^2 t+\sin^2 t=1$. This motivates the transformation:
$x(r,t)=r^3\cos^3 t$ and $y(r,t)=r^3\sin^3 t$. (Note that you can also use $x(r,t)=r\cos^3 t$ and $y(r,t)=r\sin^3 t$, but you'll have different bounds for r. We'll continue with the first transformation.)
Substituting this transformation into the astroid equation we get:

$$r^2\cos^2 t+r^2\sin^2 t=r^2=4 \Rightarrow 0\le r\le 2 \ .$$

We also have $0\le t\le 2\pi$. Now for the "fudge" factor:

$$\begin{aligned} A_{x,y}(u,v) &= \left|\left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right|\right| = \left|\left|\begin{matrix} 3r^2\cos^3 t & -3r^3\cos^2 t\sin t \\ 3r^2\sin^3 t & 3r^3\sin^2 t\cos t\end{matrix}\right|\right| \\ &= 9r^5\cos^4 t\sin^2 t+9r^5\cos^2 t\sin^4 t \\ &= 9r^5\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t) \\ &= 9r^5\cos^2 t\sin^2 t \ . \end{aligned}$$

We put this all together, so the area of the astroid is:

$$\int_0^{2\pi}\int_0^2 9r^5\cos^2 t\sin^2 t\,dr\,dt \ .$$