Name (please print):

Instructions: Show all work. No calculators, books, notes, etc.

1. What does it mean for a vector field $\{m(x, y), n(x, y)\}$ to be a gradient field? [6 points]

2. Find the gradient field of the function $f(x, y) = \sin(xy)$. [6 points]

3. Set up, but do not evaluate, the integral to compute the area between the curves $y = x^2 - 3x + 2$ and $y = -x^2 + 5x + 2$. [10 points]

4. Evaluate $\int_0^1 \int_y^1 e^{-x^2} dx dy$ by reversing the order of integration. [10 points]

5. Calculate the flow of the field $\{-y, x\}$ along the circle $x^2 + y^2 = 9$. Is the flow clockwise or counterclockwise? [8 points]

6. Calculate the flow of the field $\{x, y\}$ across the circle $x^2 + y^2 = 9$. Is the flow inward or outward? [8 points]

7. Is the flow of the vector field $\{x^4 - 6x^2y^2 + 5y^4, -4x^3y + 4xy^3\}$ suitable for modelling the flow of an incompressible fluid such as water? Why or why not? [8 points]

8. Suppose that a function $f(x, y)$ has no singularities and it's Laplacian, $\Delta f = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$. What do we know about the flow of $\nabla f(x, y)$ across any closed curve C and why? [10 points]

9. Calculate the net flow of the vector field $\{y \sin(2x), x^2 - e^{-y}\}$ along the boundary curve of the rectangle with corners at $(\pm\pi, \pm 2)$. [8 points]

$$
\begin{aligned}
&\int_{-2}^{2}\int_{-\pi}^{\pi} \operatorname{rot}\left\{y \sin(2x), x^2 - e^{-y}\right\}\, dx\, dy \\
&= \int_{-2}^{2}\int_{-\pi}^{\pi} \left(\frac{\partial(x^2 - e^{-y})}{\partial x} - \frac{\partial y \sin(2x)}{\partial y}\right) dx\, dy \\
&= \int_{-2}^{2}\int_{-\pi}^{\pi} (2x - \sin(2x))\, dx\, dy \\
&= \int_{-2}^{2} \left(x^2 + \frac{\cos(2x)}{2}\right)\Bigg|_{-\pi}^{\pi} dy = 0
\end{aligned}
$$

10. Calculate the net flow of the vector field $\left\{x^2+\frac{\sin y}{y}, e^{-x^2}-2y^2\right\}$ across the region bounded by the lines $y=-x$, $y=3-x$, $y=x/2$, and $y=x/2+4$. [10 points]

First make the transformation to a better coordinate system. Let

$$u(x,y)=x+y \text{ and } v(x,y)=y-\frac{x}{2}\ .$$

Then

$$u-v=\frac{3}{2}x \Rightarrow x(u,v)=\frac{2(u-v)}{3} \text{ and } u+2v=3y \Rightarrow y(u,v)=\frac{u+2v}{3}\ .$$

Find the "fudge-factor," i.e. the stretch factor or scaling factor.

$$A_{xy}(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{4}{9}+\frac{2}{9}=\frac{2}{3}\ .$$

Since we're calculating flow across, we find the divergence of the field:

$$\begin{aligned}\operatorname{div}\left\{x^2+\frac{\sin y}{y}, e^{-x^2}-2y^2\right\} &= \frac{\partial\left(x^2+\frac{\sin y}{y}\right)}{\partial x}+\frac{\partial\left(e^{-x^2}-2y^2\right)}{\partial y} \\ &= 2x-4y \\ &= \frac{4(u-v)}{3}-\frac{4(u+2v)}{3} \\ &= -4v\ .\end{aligned}$$

That's much nicer to work with. I guess you could say I'm a little partial to this method! HA! OK, let's put it all together and finish this.

$$\int_0^4\int_0^3(-4v)\left(\frac{2}{3}\right)\,du\,dv=\int_0^4 -8v\,dv = -4v^2\Big|_0^4=-64\ .$$

11. Set up, but do not evaluate, the integral to find the volume under $f(x,y)=e^{-((x/3)^2+y^2)}$ and over the inside and boundary of the ellipse $(x/3)^2+y^2=1$. [10 points]

First, switch coordinate systems. Note that I use r and t instead of

u and v to suggest radius and angle, which is what's going on in this case. Let

$$x(r,t) = 3r\cos t \text{ and } y(r,t) = r\sin t \ \ .$$

Then $0 \le r \le 1$ and $0 \le t \le 2\pi$ covers the entire ellipse. Get the fudge-factor.

$$A_{xy}(r,t) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 3\cos t & -3r\sin t \\ \sin t & r\cos t \end{vmatrix} = 3r\cos^t + 3r\sin^2 t = 3r \ \ .$$

Then the volume is

$$\int_0^{2\pi}\int_0^1 3re^{-r^2}\,dr\,dt \ \ .$$

This can be evaluated rather easily, but you don't need to do so.

12. How would you change the integral above to calculate the volume under $f(x,y)$ but over the entire xy−plane? [6 points]

Take r from 0 to ∞.

$$\int_0^{2\pi}\int_0^{\infty} 3re^{-r^2}\,dr\,dt$$