3.8 (SUPPLEMENT) — ORTHOGONALITY OF EIGENFUNCTIONS

We now develop some properties of eigenfunctions, to be used in Chapter 9 for Fourier Series and Partial Differential Equations.

1. Definition of Orthogonality

We say functions $f(x)$ and $g(x)$ are *orthogonal* on $a < x < b$ if $\boxed{\int_a^b f(x)g(x)\,dx = 0}$.

[Motivation: Let's approximate the integral with a Riemann sum, as follows. Take a large integer N, put $h = (b-a)/N$ and partition the interval $a < x < b$ by defining $x_1 = a + h, x_2 = a + 2h, \ldots, x_N = a + Nh = b$. Then

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\approx f(x_1)g(x_1)h + \cdots + f(x_N)g(x_N)h \\ &= (u_N \cdot v_N)h\end{aligned}$$

where $u_N = (f(x_1), \ldots, f(x_N))$ and $v_N = (g(x_1), \ldots, g(x_N))$ are vectors containing the values of f and g. The vectors u_N and v_N are said to be orthogonal (or perpendicular) if their dot product equals zero ($u_N \cdot v_N = 0$), and so when we let $N \to \infty$ in the above formula it makes sense to say the functions f and g are orthogonal when the integral $\int_a^b f(x)g(x)\,dx$ equals zero.]

Example. $\sin x$ and $\cos x$ are orthogonal on $-\pi < x < \pi$, since $\int_{-\pi}^{\pi} \sin x \cos x\,dx = \frac{1}{2}\sin^2 x\big|_{-\pi}^{\pi} = 0$.

2. Integration Lemma

Suppose functions $X_n(x)$ and $X_m(x)$ satisfy the differential equations

$$\begin{aligned} X_n'' + \lambda_n X_n &= 0, && a < x < b, \\ X_m'' + \lambda_m X_m &= 0, && a < x < b, \end{aligned}$$

for some numbers λ_n, λ_m. Then

$$(\lambda_n - \lambda_m)\int_a^b X_n(x)X_m(x)\,dx = [X_n(x)X_m'(x) - X_n'(x)X_m(x)]_a^b.$$

Proof.

$$\begin{aligned} \text{LHS} &= \int_a^b [(\lambda_n X_n)X_m - X_n(\lambda_m X_m)]\,dx && \text{by taking the } \lambda\text{'s inside the integral} \\ &= \int_a^b [-X_n'' X_m + X_n X_m'']\,dx && \text{since } \lambda_n X_n = -X_n'' \text{ and } \lambda_m X_m = -X_m'' \\ &= \int_a^b [X_n X_m' - X_n' X_m]'\,dx && \text{as you can check by differentiating!} \\ &= \text{RHS} && \text{by the Fundamental Theorem of Calculus.} \end{aligned}$$

□

3. Boundary Conditions

Consider a function $X(x)$ for $a < x < b$. Four boundary condition ("BC") types are:

$$\begin{aligned} \text{Dirichlet BC} \qquad & X(a) = X(b) = 0 \\ \text{Neumann BC} \qquad & X'(a) = X'(b) = 0 \\ \text{Mixed1 BC} \qquad & X(a) = X'(b) = 0 \\ \text{Mixed2 BC} \qquad & X'(a) = X(b) = 0 \\ \text{Periodic BC} \qquad & X(a) = X(b), \quad X'(a) = X'(b) \end{aligned}$$

(The two varieties of Mixed BC are very similar, differing only as to which endpoint has $X = 0$ and which has $X' = 0$.)

4. **Orthogonality of Eigenfunctions Theorem:**

Eigenfunctions corresponding to distinct eigenvalues must be orthogonal.

Precise statement: suppose $X_n'' + \lambda_n X_n = 0$ and $X_m'' + \lambda_m X_m = 0$ on $a < x < b$, and that X_n and X_m both satisfy the **same type of BC**. If $\lambda_n \neq \lambda_m$ then X_n and X_m are orthogonal:

$$\int_a^b X_n(x) X_m(x)\, dx = 0.$$

Proof. By the Integration Lemma, we have

$$\begin{aligned} &\int_a^b X_n(x) X_m(x)\, dx \\ &\qquad = \frac{1}{\lambda_n - \lambda_m} [X_n(x) X_m'(x) - X_n'(x) X_m(x)]_a^b \\ &\qquad = 0 \quad \text{under Dirichlet BCs, because } X_n(a) = X_n(b) = 0 \text{ and } X_m(a) = X_m(b) = 0 \\ &\qquad = 0 \quad \text{under Neumann BCs, because } X_n'(a) = X_n'(b) = 0 \text{ and } X_m'(a) = X_m'(b) = 0 \\ &\qquad = 0 \quad \text{under Mixed BCs, for similar reasons.} \end{aligned}$$

For periodic BCs, we use that $X_n(b) = X_n(a)$ and $X_m'(b) = X_m'(a)$ and so on, to see

$$\begin{aligned} [X_n(x) X_m'(x) - X_n'(x) X_m(x)]_a^b &= [X_n(b) X_m'(b) - X_n'(b) X_m(b)] \\ &\quad - [X_n(a) X_m'(a) - X_n'(a) X_m(a)] = 0. \end{aligned}$$

□

5. Example

Show that $\sin x$ and $\sin 2x$ are orthogonal for $0 < x < \pi$.

Solution. We could just show $\int_0^\pi \sin(x) \sin(2x)\, dx = 0$ by using trigonometric identities to evaluate the integral. But it is easier to notice that both $X_1(x) = \sin(x)$, with $\lambda_1 = 1^2$, and $X_2(x) = \sin(2x)$, with $\lambda_2 = 2^2$, are eigenfunctions for

$$\begin{aligned} X'' + \lambda X = 0, &\qquad 0 < x < \pi, \\ X(0) = X(\pi) = 0 &\qquad \text{(Dirichlet BC).} \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, the Orthogonality Theorem implies $\int_0^\pi X_1 X_2\, dx = 0$, so that $\sin(x)$ and $\sin(2x)$ are orthogonal for $0 < x < \pi$.

6. Orthogonality of sines and cosines for $-\pi < x < \pi$

The following formulas will be essential for our study of Fourier series, in Chapter 9.

Let $n, m \geq 1$ be integers. Then:

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = 0 \quad \text{if } n \neq m \tag{1}$$

$$= \pi \quad \text{if } n = m, \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = 0 \quad \text{if } n \neq m \tag{3}$$

$$= \pi \quad \text{if } n = m, \tag{4}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\, dx = 0 \quad \text{if } n \neq m \tag{5}$$

$$= 0 \quad \text{if } n = m. \tag{6}$$

Proof. For formula (1), we let $X_n(x) = \cos(nx)$ and $X_m(x) = \cos(mx)$, so that X_n and X_m are eigenfunctions of $X'' + \lambda X = 0$ on $-\pi < x < \pi$, with eigenvalues $\lambda_n = n^2$ and $\lambda_m = m^2$ respectively. Also, X_n and X_m satisfy **periodic** boundary conditions (since they are both 2π-periodic). If $n \neq m$ then $\lambda_n \neq \lambda_m$, and so the Orthogonality Theorem implies X_n and X_m are orthogonal, which is equation (1).

Equations (3) and (5) are proved similarly. [Exercise.]

To get equation (2), we substitute $n = m$ and evaluate the integral explicitly as

$$\int_{-\pi}^{\pi} \cos^2(mx)\, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos(2mx))\, dx = \pi.$$

Equation (4) proceeds similarly. [Exercise.] And finally, equation (6) is true since we can substitute $n = m$ and evaluate the integral explicitly as

$$\int_{-\pi}^{\pi} \cos(mx)\sin(mx)\, dx = \frac{1}{2m}\sin^2(mx)\Big|_{-\pi}^{\pi} = 0.$$

Remark. Here we have used the Orthogonality Theorem to evaluate integrals (1), (3) and (5). They can also be evaluated using trigonometric identities (or the integration formulas inside the back cover of the text). But "orthogonality of eigenfunctions" is the best way to think about these results, and we will use orthogonality in Chapter 9 for other applications as well.

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