

# Project Ib — Direction Fields\*

Richard S. Laugesen

## Goal of the project

We aim to understand graphically how properties of the function  $f(x, y)$  affect the direction field of the differential equation

$$\frac{dy}{dx} = f(x, y).$$

## Introduction

We need the concept of periodicity. A function  $g(x)$  is called **periodic** if for some number  $P > 0$  one has

$$g(x) = g(x + P)$$

for all  $x$ . Then  $P$  is called a *period* of the function  $g$ .

*Example 1.*  $g(x) = \sin(x)$  is periodic with  $P = 2\pi$ , since  $\sin(x) = \sin(x + 2\pi)$  for all  $x$ . Notice that the graph of  $\sin x$  repeats itself every  $2\pi$  units, which we expect from the periodicity.

*Example 2.*  $g(x) = \sin(x) + 1$  is periodic with  $P = 2\pi$ , since  $\sin(x) + 1 = \sin(x + 2\pi) + 1$  for all  $x$ .

*Example 3.* The function  $g(x)$  defined piecewise by the rule

$$g(x) = \begin{cases} 1, & \text{if } -2 \leq x < -1 \text{ or } 0 \leq x < 1 \text{ or } 2 \leq x < 3, \dots, \\ 2, & \text{if } -1 \leq x < 0 \text{ or } 1 \leq x < 2 \text{ or } 3 \leq x < 4, \dots \end{cases}$$

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is periodic with  $P = 2$ . (Exercise: sketch a graph of this function!) This example shows that periodic functions need not be trigonometric.

*Example 4.*  $g(x, y) = 3^{\cos 2x} + y^2$ . This function of two variables is **periodic in the  $x$ -variable** with period  $P = \pi$ , because  $g(x, y) = g(x + \pi, y)$  for all  $x$  and  $y$ . But  $g(x, y)$  is not periodic in the  $y$ -variable, because no matter what  $P > 0$  you try, you can check that  $g(x, 0) \neq g(x, P)$ , for example.

## Problems

Give both an Answer and an Explanation, for each problem below. A sample solution is provided for the first question, in order to show the level of detail required.

Do not expect to answer these questions just by looking at them! Mathematics is an experimental science, and **experimentation with Examples 1–4 is essential** for answering Questions 3–5. That is, in order to get ideas you should use Iode to plot direction fields involving the periodic functions in Examples 1–4, and then experiment by modifying those examples. For Questions 6 and 7, you can experiment with the direction fields you used in Project Ia.

Include your Iode plots with your solution, if you feel the plots help explain your answer.

1. The solution curves of  $\frac{dy}{dx} = f(x, y)$  are all increasing from left to right if [fill in the blanks].

*Answer:* ... if  $f(x, y) > 0$  for all  $x$  and  $y$ , in other words if  $f$  is positive everywhere.

*Explanation:* If  $f$  is positive then  $\frac{dy}{dx}$  is positive, which means the slope is positive and so the curve is increasing from left to right. For example, you can see this happening in the direction field of  $dy/dx = y^2 + 1$  (which you could plot and attach to your solution); in that example,  $f(x, y) = y^2 + 1 > 0$  for all  $x, y$ .

2. Suppose  $y = y(x)$  solves  $\frac{dy}{dx} = f(x)$ , that is,  $y'(x) = f(x)$ .  
If  $y(x)$  is periodic then  $f(x)$  is periodic. (True/False, and Explain)
3. Suppose  $y = y(x)$  solves  $\frac{dy}{dx} = f(x)$ .  
If  $f(x)$  is periodic then  $y(x)$  is periodic. (True/False, and Explain)

4. The plot of the direction field for  $\frac{dy}{dx} = f(x, y)$  shows a vertically repeating pattern (e.g. Fig. E in Project Ia) if [fill in the blanks, and Explain].
5. Suppose  $y = y(x)$  solves  $\frac{dy}{dx} = f(x)$ .  
If  $y(x) \rightarrow \infty$  as  $x \rightarrow \infty$  then  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . (True/False, and Explain)
6. Consider  $\frac{dy}{dx} = f(y)$ , and suppose  $f(2) = 0$ . What feature do you observe in the direction field, at height 2?  
(*Note.* Here  $f$  does not depend on  $x$ .)
7. Find a function  $f(y)$  with the property that: for each solution  $y(x)$  of  $\frac{dy}{dx} = f(y)$ , the limiting value  $\lim_{x \rightarrow \infty} y(x)$  equals 3 if  $y(0) > 0$  but not if  $y(0) < 0$ . [Your explanation can consist of a direction field with illustrative solution curves plotted on it.]

## Reference: mathematical expressions in Matlab, Octave

For simple expressions, we use the usual keyboard characters:

`2*x` means  $2x$ ,  
`(x^3-1)/6` means  $(x^3 - 1)/6$ ,  
`pi` means  $\pi$ .

### Built-in functions

<code>exp(x)</code>	exponential, $e^x$		
<code>log(x)</code>	natural logarithm, $\ln x$		
<code>log10(x)</code>	base 10 logarithm, $\log_{10} x$		
<code>abs(x)</code>	absolute value, $ x $		
<code>sqrt(x)</code>	square root, $\sqrt{x}$		
<code>sign(x)</code>	signum function, which equals	$\begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$	
<code>sin(x)</code>		<code>sinh(x)</code>	
<code>cos(x)</code>	trigonometric	<code>cosh(x)</code>	hyperbolic
<code>tan(x)</code>	functions	<code>tanh(x)</code>	trigonometric
<code>cot(x)</code>	( $x$ in radians)	<code>coth(x)</code>	functions
<code>sec(x)</code>		<code>sech(x)</code>	
<code>csc(x)</code>		<code>csch(x)</code>	
<code>asin(x)</code>		<code>asinh(x)</code>	
<code>acos(x)</code>	inverse	<code>acosh(x)</code>	inverse
<code>atan(x)</code>	trigonometric	<code>atanh(x)</code>	hyperbolic
<code>acot(x)</code>	functions	<code>acoth(x)</code>	trigonometric
<code>asec(x)</code>		<code>asech(x)</code>	functions
<code>acsc(x)</code>		<code>acsch(x)</code>	

#### Example 1.

`sin(exp(y))^4` means  $\sin^4(e^y)$ ,  
`acos(exp(1)^(-1))` means  $\arccos(e^{-1})$ .

More Matlab and Octave commands are explained at [www.octave.org/docs.html](http://www.octave.org/docs.html).