

## UNDETERMINED COEFFICIENTS — SUMMARY SHEET

**Goal:** to solve  $\boxed{Ly = f}$  where  $L$  is a constant coefficient linear differential operator and

$$\begin{aligned} f(x) &= P_m(x)e^{rx} \cos(kx) \\ \text{or} \quad &= P_m(x)e^{rx} \sin(kx) \\ \text{or} \quad &= P_m(x)e^{rx} \end{aligned}$$

where  $P_m$  is some polynomial of degree  $m \geq 0$ , and  $r \in \mathbb{R}$  and  $k > 0$  are constants. (If  $r = 0$  then the exponential  $e^{rx}$  can be ignored.)

**First find the complementary solution**  $y_c(x)$  (the solution of  $Ly = 0$ ). Then...

### Rule 1

To find a particular solution  $y_p$  of  $Ly = f$ , try guessing

$$\begin{aligned} y_p(x) &= [A_0 + A_1x + \cdots + A_mx^m]e^{rx} \cos(kx) \\ &\quad + [B_0 + B_1x + \cdots + B_mx^m]e^{rx} \sin(kx) \end{aligned}$$

or just  $y_p(x) = [A_0 + A_1x + \cdots + A_mx^m]e^{rx}$  if there are no sin or cos terms in  $f(x)$ .

If **no term** in the guess for  $y_p$  **duplicates** a term in the complementary solution  $y_c$ , then substitute the guess into  $Ly = f$  and determine the coefficients  $A_0, A_1, \dots, A_m$  and  $B_0, B_1, \dots, B_m$ . Then  $\boxed{y = y_c + y_p}$  is the general solution of  $Ly = f$ .

### Rule 2

But if some terms *are* duplicated, change the guess to

$$\begin{aligned} y_p(x) &= x^s[A_0 + A_1x + \cdots + A_mx^m]e^{rx} \cos(kx) \\ &\quad + x^s[B_0 + B_1x + \cdots + B_mx^m]e^{rx} \sin(kx) \end{aligned}$$

or just  $y_p(x) = x^s[A_0 + A_1x + \cdots + A_mx^m]e^{rx}$  if there are no sin or cos terms in  $f(x)$ ,

where  $s$  is the smallest exponent that succeeds in eliminating duplication with  $y_c$ . That is, keep multiplying through your original guess by powers of  $x$  until you have eliminated all duplication.

Then substitute into  $Ly = f$  and determine the coefficients  $A_0, A_1, \dots, A_m$  and  $B_0, B_1, \dots, B_m$ . Then  $\boxed{y = y_c + y_p}$  is the general solution of  $Ly = f$ .

*Note.* Figure 3.5.1 in the textbook summarizes the most common guesses for  $y_p$ .

**What if  $f$  consists of more than one term?** In that case, find  $y_p$  for each term separately, then **add the  $y_p$ 's**.

For example, if  $f(x) = x^2 \sin(x) + e^{4x}$ , then we should use Rules 1 and 2 to find a particular solution  $y_{p,1}(x)$  corresponding to  $f_1(x) = x^2 \sin(x)$ , and then similarly find a particular solution  $y_{p,2}(x)$  corresponding to  $f_2(x) = e^{4x}$ . Then  $y_p = y_{p,1} + y_{p,2}$  will be a particular solution of our original problem.

(This is just an application of the Superposition Principle for nonhomogeneous linear equations.)

**Example 1.**  $y'' + 2y = 4e^{3x}$

**Example 2.**  $y'' - 3y' + 2y = x^2 + x - 5$

**Example 3.**  $2y'' - y' + y = 3 \sin(2x)$

**Example 4.**  $y'' + 9y = \cos(3x)$

**Example 5.**  $y'' + 6y' + 16y = e^{-3x} \sin(\sqrt{7}x)$

**The annihilator explanation for Rule 2 (duplication):**

Optional. See the *Annihilators* handout on the class website.