NAME:

Math 285 Spring 2003 — Final Exam

Total points: **200**. **Do only two of #9, 10, 11, 12.** (Cross out the two you are not doing.) Explain all answers. No notes, books, calculators or computers.

1. [20 points] Solve

$$y' - 2xy = -e^{x-x^2} y^2, \qquad y(0) = \frac{1}{2}.$$

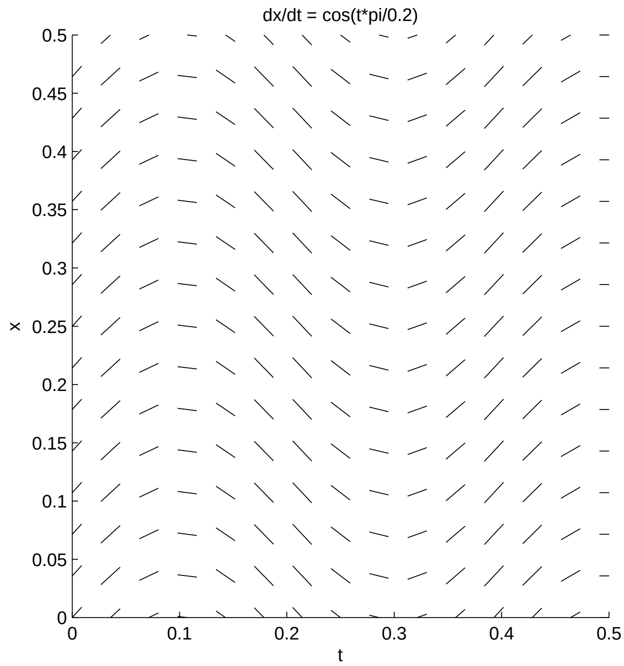
2. [20 points] Your car loan must be paid off over 60 months, at a monthly rate of 1% with monthly payments of \$100, so that

$$\frac{dP}{dt} = (0.01)P - 100$$

where $P(t)$ is the principal (amount still owed) after t months.

Find the amount you can afford to borrow. That is, find $P(0)$.

Hint. $e^{-0.6} \approx 0.55$



3. [15=5+10 points] The Euler update formula for $\frac{dx}{dt} = f(t, x)$ says that $\boxed{x_{i+1} = x_i + hk_1}$ where $k_1 = f(t_i, x_i)$.

(a) What does the Improved Euler update formula say?

$\boxed{x_{i+1} = \qquad\qquad\qquad\qquad}$

(b) On the direction field above, take $t_0 = 0, x_0 = 0.25$ and

(i) sketch the solution curve,

(ii) sketch the first step of the Euler method with $h = 0.1$,

(iii) sketch the first step of the Improved Euler method with $h = 0.1$.

Label each sketch clearly.

4. [15 points] Consider the damped oscillator equation $mx'' + cx' + kx = 0$. Suppose the oscillator is underdamped and the amplitude of the solution decreases by 70% for each time unit that passes by (*e.g.* from $t = 0$ to $t = 1$). Show that

$$c = 2m \log \frac{10}{3}, \qquad \text{and} \qquad k > m \left(\log \frac{10}{3} \right)^2 .$$

5. [30=18+6+6 points] (a) Show by using Undetermined Coefficients that a particular solution of the damped, forced oscillator equation

$$x''(t) + 2x'(t) + 4x(t) = \cos(\omega t) \tag{1}$$

is

$$x_p(t) = \frac{4 - \omega^2}{(4 - \omega^2)^2 + (2\omega)^2} \cos(\omega t) + \frac{2\omega}{(4 - \omega^2)^2 + (2\omega)^2} \sin(\omega t).$$

5. (b) The solution in (a) can also be written $x_p(t) = C(\omega)\cos(\omega t - \gamma)$ where

$$C(\omega) = \frac{1}{\sqrt{(4-\omega^2)^2 + (2\omega)^2}}.$$

Show that practical resonance occurs at $\omega = \sqrt{2}$.

5. (c) Sketch the solution $x_c + x_p$ of equation (1) for large t, assuming $\omega = \sqrt{2}$ and $x(0) = 10, x'(0) = 0$.

6. [20=4+12+4 points] Let

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t < 0, \\ t & \text{if } 0 < t < \pi, \end{cases}$$

and extend f to be 2π-periodic.

(a) Sketch the graph of f over two periods. On the same graph, roughly sketch the 25^{th} partial sum of f.

(b) Show that the Fourier coefficients of f are

$$A_0 = \frac{\pi}{2}, \qquad A_n = \frac{(-1)^n - 1}{\pi n^2}, \qquad B_n = -\frac{(-1)^n}{n}.$$

(c) Use the Fourier series $\frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(nt) + B_n \sin(nt)]$ of f, at $t = \pi$, to show that $\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$.

7. [25=3+22 points] (Periodically forced, undamped oscillator.) Consider

$$x''(t) + 16x(t) = f(t)$$

where $f(t)$ is the 2π-periodic function from problem #6.

(a) Find the complementary solution.

(b) Find a particular solution. (You can use the result of #6.)

8. [15 points] The heat equation $u_t = u_{xx}$ (with $k = 1$) on $0 < x < \frac{\pi}{2}$ with mixed boundary conditions $u(0,t) = 0$ and $u_x(\frac{\pi}{2}, t) = 0$ has solution

$$u(x,t) = \sum_{n \text{ odd}} c_n e^{-n^2 t} \sin(nx).$$

Suppose $u(x,0) = e \sin(x) + 100 \sin(5x)$. Find the solution $u(x,t)$ and sketch it at $t = 0$ and at $t = 1$.

9. [20 points] *(Do only two of #9, 10, 11, 12.)* Consider the eigenvalue problem $X''(x) + \lambda X(x)$ for $0 < x < \pi$, with the boundary conditions

$$X(0) = 0, \qquad X(\pi) - X'(\pi) = 0.$$

[We did not study these boundary conditions in class.] Show that either

$$\lambda > 0 \qquad \text{and} \qquad \boxed{\tan(\sqrt{\lambda}\pi) = \sqrt{\lambda}}$$

or else

$$\lambda < 0 \qquad \text{and} \qquad \boxed{\tanh(\sqrt{|\lambda|}\pi) = \sqrt{|\lambda|}}.$$

Aside. The eigenvalues can then be found from these formulas, either graphically or numerically.

10. [20 points] *(Do only two of #9, 10, 11, 12.)* Use separation of variables to find a series solution of the wave equation with Dirichlet boundary conditions:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < L, \\ u(0,t) = u(L,t) &= 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

Your answer must evaluate the coefficients in terms of f and g.

Note. You may use that the Dirichlet eigenfunctions are $X_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$.

11. [20=10+6+4 points] *(Do only two of #9, 10, 11, 12.)* Consider

$$u_{tt} = 2^2 u_{xx},$$
$$u(x,0) = \begin{cases} 2 - 2x^2 & \text{if } -1 \le x \le 1, \\ 0 & \text{if } x < -1 \text{ or } x > 1, \end{cases}$$
$$u_t(x,0) = 0.$$

(a) Suppose we are working on the whole line $-\infty < x < \infty$. Sketch the solution at $t = 0$, and at $t = 2$. *Hint.* D'Alembert.

(b) Suppose we are working on the interval $-3 < x < 3$, with Neumann boundary conditions. Sketch the solution at $t = 0$, and at $t = 2$. (You may use observations from the final Iode homework.)

(c) If we found a series solution for the problem in part (b), then roughly how many terms of the series would be needed to get "naked eye convergence"? Explain.

12. [20=16+4 points] *(Do only two of #9, 10, 11, 12.)* Consider Laplace's equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < L, \quad 0 < y < L, \\ u &= 0 \qquad \text{when } y = 0 \text{ or } y = L \text{ or } x = L. \end{aligned}$$

(a) Use separation of variables to find the solution

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(x - L)}{L}\right) \sin\left(\frac{n\pi y}{L}\right).$$

(b) Find the solution with $u(0, y) = \sin(\frac{2\pi y}{L})$.

Formulas

Here are some formulas you might be able to use on the exam:

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}, \quad \omega_1 = \sqrt{\omega_0^2 - p^2} \\
e^{(a \pm ib)x} &= e^{ax}(\cos bx \pm i \sin bx) \\
y &= -y_1 \int \frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx \\
W &= y_1 y_2' - y_1' y_2 \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \\
\int u \cos u\, du &= u \sin u + \cos u + C \\
\int u \sin u\, du &= -u \cos u + \sin u + C \\
u(x,t) &= X(x)T(t)
\end{aligned}
$$