Math 285 Spring 2003 — Final Exam Answers

Total points: **200**. **Do only two of #9, 10, 11, 12.** (Cross out the two you are not doing.) Explain all answers. No notes, books, calculators or computers.

1. [20 points] Solve

$$y' - 2xy = -e^{x-x^2} y^2, \qquad y(0) = \frac{1}{2}.$$

Solution. Make the Bernoulli substitution $v = y^{1-2} = y^{-1}$, which leads to the first order linear equation $v' + 2xv = e^{x-x^2}$. By the integrating factor method one solves to get v, and then finds

$$y = \frac{e^{x^2}}{e^x + 1}.$$

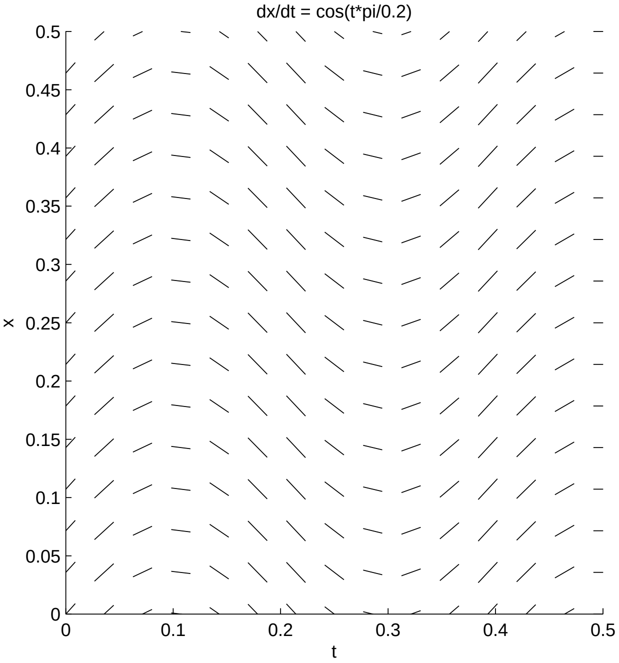
2. [20 points] Your car loan must be paid off over 60 months, at a monthly rate of 1% with monthly payments of \$100, so that

$$\frac{dP}{dt} = (0.01)P - 100$$

where $P(t)$ is the principal (amount still owed) after t months.

Find the amount you can afford to borrow. That is, find $P(0)$.
Hint. $e^{-0.6} \approx 0.55$

Solution. The equation is first order linear (it is also separable, but the first order linear method is more reliable). One finds $P(t) = 10000 + Ce^{.01t}$. Determine $C = -5500$ by using $P(60) = 0$, then evaluate $P(0) = 4500$ dollars.



3. [15=5+10 points] The Euler update formula for $\frac{dx}{dt} = f(t, x)$ says that $\boxed{x_{i+1} = x_i + hk_1}$ where $k_1 = f(t_i, x_i)$.

(a) What does the Improved Euler update formula say?

$\boxed{x_{i+1} = x_i + h(k_1 + k_2)/2}$ where $k_2 = f(t_i + h, x_i + hk_1)$ is the slope at the next "Euler" point

(b) On the direction field above, take $t_0 = 0, x_0 = 0.25$ and

(i) sketch the solution curve,

(ii) sketch the first step of the Euler method with $h = 0.1$, $\boxed{t_1 = 0.1, x_1 = 0.35}$

(iii) sketch the first step of the Improved Euler method with $h = 0.1$. $\boxed{t_1 = 0.1, x_1 = 0.30}$

Label each sketch clearly.

4. [15 points] Consider the damped oscillator equation $mx'' + cx' + kx = 0$. Suppose the oscillator is underdamped and the amplitude of the solution decreases by 70% for each time unit that passes by (*e.g.* from $t = 0$ to $t = 1$). Show that

$$c = 2m \log \frac{10}{3}, \qquad \text{and} \qquad k > m \left(\log \frac{10}{3} \right)^2 .$$

Solution. We know from class that an underdamped oscillator has time-varying amplitude Ce^{-pt}. So we want

$$Ce^{-p \cdot 1} = (30\%) Ce^{-p \cdot 0}$$

or $e^{-p} = 3/10$, that is $p = \log \frac{10}{3}$. Now recall $p = c/2m$.

Finally, the condition for underdamping is $c^2 < 4km$, and so we just substitute $c = 2m \log \frac{10}{3}$ into this inequality and simplify.

5. [30=18+6+6 points] (a) Show by using Undetermined Coefficients that a particular solution of the damped, forced oscillator equation

$$x''(t) + 2x'(t) + 4x(t) = \cos(\omega t) \tag{1}$$

is

$$x_p(t) = \frac{4 - \omega^2}{(4 - \omega^2)^2 + (2\omega)^2} \cos(\omega t) + \frac{2\omega}{(4 - \omega^2)^2 + (2\omega)^2} \sin(\omega t).$$

Solution. Guess $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$. [$x_p$ does not duplicate x_c, because of the damping, which causes x_c to contain an exponential decay factor.]

Subsitute x_p into the DE, and equate coefficients of cos and sin to get a pair of equations for A and B. Solving these equations gives the above formula for x_p.

5. (b) The solution in (a) can also be written $x_p(t) = C(\omega)\cos(\omega t - \gamma)$ where

$$C(\omega) = \frac{1}{\sqrt{(4-\omega^2)^2 + (2\omega)^2}}.$$

Show that practical resonance occurs at $\omega = \sqrt{2}$.

Solution. Practical resonance occurs when $C(\omega)$ is maximal. According to the graph provided, this will be when $C'(\omega) = 0$. Carrying out the differentiation leads to $\omega = \sqrt{2}$.

5. (c) Sketch the solution $x_c + x_p$ of equation (1) for large t, assuming $\omega = \sqrt{2}$ and $x(0) = 10, x'(0) = 0$.

Solution. $x_c(t)$ is transient, and so for large t the solution is dominated by $x_p(t)$, which oscillates with period $2\pi/\omega = 2\pi/\sqrt{2}$ and amplitude $C(\sqrt{2}) = 1/\sqrt{12}$.

6. [20=4+12+4 points] Let

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t < 0, \\ t & \text{if } 0 < t < \pi, \end{cases}$$

and extend f to be 2π-periodic.

(a) Sketch the graph of f over two periods. On the same graph, roughly sketch the 25th partial sum of f.

Solution. The Fourier partial sum graph should show some overshoot at the jump points.

(b) Show that the Fourier coefficients of f are

$$A_0 = \frac{\pi}{2}, \qquad A_n = \frac{(-1)^n - 1}{\pi n^2}, \qquad B_n = -\frac{(-1)^n}{n}.$$

Solution. Use the usual formulas for A_0, A_n, B_n, then change variable with $u = nt$ and $du = n\,dt$, and just apply the integration formulas provided on the formula sheet.

(c) Use the Fourier series $\frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(nt) + B_n \sin(nt)]$ of f, at $t = \pi$, to show that $\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$.

Solution. Plug $t = \pi$ into the Fourier series and recall $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$. The most important point is then that this Fourier series equals the average of the values of f from the left and right of the jump point, at $t = \pi$:

$$\frac{f(\pi-) + f(\pi+)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.$$

7. [25=3+22 points] (Periodically forced, undamped oscillator.) Consider

$$x''(t) + 16x(t) = f(t)$$

where $f(t)$ is the 2π-periodic function from problem #6.

(a) Find the complementary solution.

Solution. $x_c = c_1 \cos(4t) + c_2 \sin(4t)$

(b) Find a particular solution. (You can use the result of #6.)

Solution. Since $f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(nt) + B_n \sin(nt)]$ by #6, we guess

$$x_p(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)]$$

for some numbers a_0, a_n, b_n to be determined. But we notice that the terms in x_p having $n = 4$ duplicate the complementary solution x_c. So we multiply those terms by t, guessing

$$x_p(t) = \frac{1}{2}a_0 + t[a_4 \cos(4t) + b_4 \sin(4t)] + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty}[a_n \cos(nt) + b_n \sin(nt)].$$

We substitute this guess into the DE $x''(t) + 16x(t) = f(t)$ and equate coefficients of the constant terms and the $\cos(4t), \sin(4t), \cos(nt), \sin(nt)$ terms to arrive at

$$a_0 = \frac{\pi}{32}, \quad a_4 = \frac{1}{32}, \quad b_4 = 0, \quad a_n = \frac{(-1)^n - 1}{(16 - n^2)\pi n^2}, \quad b_n = \frac{(-1)^{n+1}}{(16 - n^2)n}.$$

8. [15 points] The heat equation $u_t = u_{xx}$ (with $k = 1$) on $0 < x < \frac{\pi}{2}$ with mixed boundary conditions $u(0,t) = 0$ and $u_x(\frac{\pi}{2}, t) = 0$ has solution

$$u(x,t) = \sum_{n \text{ odd}} c_n e^{-n^2 t} \sin(nx).$$

Suppose $u(x,0) = e\sin(x) + 100\sin(5x)$. Find the solution $u(x,t)$ and sketch it at $t = 0$ and at $t = 1$.

Solution.

$$\begin{aligned} u(x,t) &= ee^{-1^2 t}\sin(x) + 100e^{-5^2 t}\sin(5x) \\ &= e^{1-t}\sin(x) + 100e^{-25t}\sin(5x) \end{aligned}$$

Clearly

$$u(x,0) = e\sin(x) + 100\sin(5x)$$

is dominated by $\boxed{100\sin(5x)}$, while

$$u(x,1) = \sin(x) + 100e^{-25}\sin(5x)$$

is dominated by $\boxed{\sin(x)}$. We sketch the dominant parts...

9. [20 points] *(Do only two of #9, 10, 11, 12.)* Consider the eigenvalue problem $X''(x) + \lambda X(x)$ for $0 < x < \pi$, with the boundary conditions

$$X(0) = 0, \qquad X(\pi) - X'(\pi) = 0.$$

[We did not study these boundary conditions in class.] Show that either

$$\lambda > 0 \qquad \text{and} \qquad \boxed{\tan(\sqrt{\lambda}\pi) = \sqrt{\lambda}}$$

or else

$$\lambda < 0 \qquad \text{and} \qquad \boxed{\tanh(\sqrt{|\lambda|}\pi) = \sqrt{|\lambda|}}.$$

Aside. The eigenvalues can then be found from these formulas, either graphically or numerically.

Solution. If $\lambda > 0$ then the general solution is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

The boundary condition $X(0) = 0$ implies $A = 0$, so that $X(x) = B\sin(\sqrt{\lambda}x)$. The other condition $X(\pi) - X'(\pi) = 0$ now gives

$$B[\sin(\sqrt{\lambda}\pi) - \sqrt{\lambda}\cos(\sqrt{\lambda}\pi)] = 0.$$

If $B = 0$ then $X(x) \equiv 0$, and so for λ to be an eigenvalue we need $\sin(\sqrt{\lambda}\pi) - \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$, which rearranges to $\tan(\sqrt{\lambda}\pi) = \sqrt{\lambda}$.

If $\lambda < 0$ then argue similarly, from the general solution

$$X(x) = A\cosh(\sqrt{|\lambda|}x) + B\sinh(\sqrt{|\lambda|}x).$$

Finally, if $\lambda = 0$ then the general solution is $X(x) = A + Bx$. The boundary condition $X(0) = 0$ implies $A = 0$, so that $X(x) = Bx$. The other condition $X(\pi) - X'(\pi) = 0$ now gives $B(\pi - 1) = 0$, and so $B = 0$. That is $X(x) \equiv 0$, meaning $\lambda = 0$ is not an eigenvalue.

10. [20 points] *(Do only two of #9, 10, 11, 12.)* Use separation of variables to find a series solution of the wave equation with Dirichlet boundary conditions:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad 0 < x < L, \\ u(0,t) = u(L,t) &= 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

Your answer must evaluate the coefficients in terms of f and g.

Note. You may use that the Dirichlet eigenfunctions are $X_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$.

Solution. Substitute $u(x,t) = X(x)T(t)$ into the wave equation $u_{tt} = c^2 u_{xx}$, to arrive at

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda.$$

So

$$\begin{aligned} T'' + c^2 \lambda T &= 0 \\ X'' + \lambda X &= 0 \\ X(0) = X(L) &= 0 \end{aligned}$$

where the equations $X(0) = X(L) = 0$ come from the boundary conditions $u(0,t) = u(L,t) = 0$. Hence (from Section 3.8) $X_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$ with eigenvalue $\lambda_n = (n\pi/L)^2$.

Then $T_n(t) = c_n \cos(n\pi ct/L) + d_n \sin(n\pi ct/L)$.

Now we add up all the separated solutions:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} [c_n \cos(n\pi ct/L) + d_n \sin(n\pi ct/L)] \sin(n\pi x/L).$$

Finally we apply the initial conditions:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$$

and so c_n is the nth sine coefficient of f, meaning $c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$. And

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} d_n(n\pi c/L)\sin(n\pi x/L)$$

and so $d_n(n\pi c/L)$ is the nth sine coefficient of g, meaning $d_n = (L/n\pi c)\frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)\,dx$.

11. [20=10+6+4 points] *(Do only two of #9, 10, 11, 12.)* Consider

$$\begin{aligned} u_{tt} &= 2^2 u_{xx}, \\ u(x,0) &= \begin{cases} 2-2x^2 & \text{if } -1 \le x \le 1, \\ 0 & \text{if } x < -1 \text{ or } x > 1, \end{cases} \\ u_t(x,0) &= 0. \end{aligned}$$

(a) Suppose we are working on the whole line $-\infty < x < \infty$. Sketch the solution at $t = 0$, and at $t = 2$. *Hint.* D'Alembert.

Solution. Since the string is being released from rest, the D'Alembert solution tells us that half the displacement will move left and half will move right, each with speed $c = 2$.

(b) Suppose we are working on the interval $-3 < x < 3$, with Neumann boundary conditions. Sketch the solution at $t = 0$, and at $t = 2$. (You may use observations from the final Iode homework.)

Solution. Again half the displacement moves left and half right. But this time the waves get reflected back at the endpoints.

Because the boundary conditions are Neumann, the waves do *not* get flipped upside down.

(c) If we found a series solution for the problem in part (b), then roughly how many terms of the series would be needed to get "naked eye convergence"? Explain.

Solution. Around 15-30 terms would be needed, because the waves have corners but no jumps.

12. [20=16+4 points] *(Do only two of #9, 10, 11, 12.)* Consider Laplace's equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < L, \quad 0 < y < L, \\ u &= 0 \qquad \text{when } y = 0 \text{ or } y = L \text{ or } x = L. \end{aligned}$$

(a) Use separation of variables to find the solution

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(x-L)}{L}\right) \sin\left(\frac{n\pi y}{L}\right).$$

Solution. This is very similar to an Example in Section 9.7 of the textbook.

(b) Find the solution with $u(0,y) = \sin(\frac{2\pi y}{L})$.

Solution. Only the $n = 2$ term of the solution is needed:

$$u(x,y) = \frac{1}{\sinh(-2\pi)} \sinh(2\pi(x-L)/L)\sin(2\pi y/L)$$

Formulas

Here are some formulas you might be able to use on the exam:

$$\begin{aligned}
\omega_0 &= \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}, \quad \omega_1 = \sqrt{\omega_0^2 - p^2} \\
e^{(a \pm ib)x} &= e^{ax}(\cos bx \pm i \sin bx) \\
y &= -y_1 \int \frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx \\
W &= y_1 y_2' - y_1' y_2 \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \\
\int u \cos u\, du &= u \sin u + \cos u + C \\
\int u \sin u\, du &= -u \cos u + \sin u + C \\
u(x,t) &= X(x)T(t)
\end{aligned}$$