Row operations to solve a vector equation

Consider a vector equation $A\vec{c} = \vec{b}$ where A is a matrix of constants, $\vec{b}$ is a vector of constants, and $\vec{c}$ is the unknown vector for which we wish to solve. Writing out the equation $A\vec{c} = \vec{b}$ in full detail:

$$\begin{aligned} a_{11}c_1 + \cdots + a_{1n}c_n &= b_1 \\ &\vdots \\ a_{n1}c_1 + \cdots + a_{nn}c_n &= b_n \end{aligned}$$

We can represent the equations with the *augmented coefficient matrix*

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & b_n \end{bmatrix}$$

It is permissible to

(1) multiply any row (that is, equation) by a nonzero constant,
(2) interchange any two rows (that is, equations),
(3) add or subtract a constant multiple of any row (that is, equation) from any other row (that is, equation).

The goal is to perform these *row operations* until the augmented matrix is “upper triangular”, meaning it has zeros below the main diagonal. (We achieve this by concentrating on the first column, and then the second column, and so on.) Then we perform more row operations to get zeros above the main diagonal too, if possible. Finally we write out the resulting equations, and solve for $c_n, c_{n-1}, \dots, c_1$.

Example. Solve

$$\begin{aligned} 3c_1 - c_2 + 2c_3 &= 1 \\ 2c_2 - 5c_3 &= 2 \\ c_1 + 4c_2 - c_3 &= -2 \end{aligned}$$