Math 285 Spring 2003 — Test 1 Solutions

Total points: **75**. Do all questions. Explain all answers.No notes, books, calculators or computers.

1. [12=5+7 points] A flu epidemic hits Chicago. Write C for the total number of people in Chicago, and $N(t)$ for the number of people who are sick with the flu on the t^{th} day of the epidemic. Assume that each day, $\frac{1}{10}$ of the healthy people become sick and $\frac{1}{8}$ of the sick people become healthy again.

(a) Write down a differential equation involving $N(t)$. (Don't solve it.)
Note. Assume it is possible for a person to catch the flu more than once.

Solution. **(Similar to 1.1 #35, 36.)**

$$\# \text{ sick people on day } t = N(t)$$
$$\# \text{ healthy people on day } t = C - N(t)$$

so the problem says

$$\frac{dN}{dt} = \frac{1}{10}(C - N) - \frac{1}{8}N.$$

(b) What proportion of Chicagoans will be sick with the flu, after the epidemic has had time to spread thoroughly?
Hint. Phase line.

Solution. The differential equation can be simplified to

$$\frac{dN}{dt} = \frac{1}{10}\left(C - \frac{18}{8}N\right).$$

To find the equilibrium point we set the righthand side equal to zero: this happens when $N = \frac{8}{18}C$. This equilibrium is stable because $C - \frac{18}{8}N$ is negative when N is larger than $\frac{8}{18}C$, and is positive when N is smaller than $\frac{8}{18}C$. (See phase line at left.) Since the equilibrium at $N = \frac{8}{18}C$ is stable, it is reasonable to guess that $\frac{8}{18}$ of the population will have the flu, when the epidemic has had time to spread through the city.

Aside. Of course, this model is unrealistic because once you have had a partiuclar strain of the flu, you become immune and cannot catch it again. Can you find a differential equation that takes account of this immunity?

2. [4 points] Find an example of a function f for which the direction field of $\frac{dy}{dx} = f(x, y)$ shows a *vertically* repeating pattern. Explain.

Solution. As we saw in **Project 1 #10**, a vertically repeating pattern in the direction field is caused by a function f that is periodic in the y-variable, meaning $f(x, y) = f(x, y + P)$ for all $x, y \in \mathbb{R}$. For example, the direction field of

$$f(x, y) = x^2 + \cos(3y)$$

would have a pattern repeating every $\frac{2\pi}{3}$ units vertically, because $\cos(3y)$ has period $\frac{2\pi}{3}$.

3. [4 points] True/False, and Explain: if the direction field for $\frac{dy}{dx} = f(x, y)$ shows a *horizontally* repeating pattern, then each solution $y(x)$ must be periodic.

Solution. False. For example, the direction field of

$$f(x, y) = 1 + \cos(x)$$

has a horizontally repeating pattern, since f is periodic in the x-variable. But as we saw in **Project 1 #9**, the solution of $\frac{dy}{dx} = 1 + \cos(x)$ is

$$y(x) = x + \sin(x) + C,$$

which is certainly not periodic because of the "x" term that tends to infinity as $x \to \infty$.

4. [20=14+2+4 points] Write $x(t)$ for the height at time t of an object that is falling downward under the influence of gravity. Assume the object encounters air resistance proportional to the square of its velocity $v(t) = x'(t)$. Then by Newton's Law we get

$$\frac{dv}{dt} = av^2 - g \qquad \text{(for some positive constants } a \text{ and } g\text{)}.$$

(a) Solve for $v(t)$. *Hint.* Write $b = \sqrt{g/a}$, to simplify the calculations.

Solution. The equation is **separable** (note: it is *not* linear, due to the v^2, and it is *not* of Bernoulli type, due to the $-g$).

$$\begin{aligned}
\frac{dv}{a(v^2 - g/a)} &= dt \\
-\int \frac{dv}{b^2 - v^2} &= \int a\,dt && \text{since } b^2 = g/a \\
-\frac{1}{2b}\log\left|\frac{v+b}{v-b}\right| &= at + C && \text{by the integral written on the board} \\
\frac{v+b}{v-b} &= \pm e^{-2bC}e^{-2abt} = Ae^{-2abt} && \text{where } A = \pm e^{-2bC} \\
v + b &= (v-b)Ae^{-2abt} \\
v(1 - Ae^{-2abt}) &= -b(1 + Ae^{-2abt}) \\
v &= -b\frac{1 + Ae^{-2abt}}{1 - Ae^{-2abt}}
\end{aligned}$$

(b) Find the terminal velocity.

Solution. The terminal velocity is

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -b\frac{1+0}{1-0} = -b.$$

Notice the terminal velocity must be *negative*, since the object is *falling*.

(c) Sketch the phase line for v. Does it agree with your answer to (b)?

Solution. The phase line of $\frac{dv}{dt} = av^2 - g$ has an equilibrium point when the righthand side equals zero: $\boxed{av^2 - g = 0}$, or $v = \pm\sqrt{g/a} = \pm b$. Notice that $av^2 - g$ is positive ("up arrow" on the phase line) except for $-\sqrt{g/a} < v < \sqrt{g/a}$, in which case it is negative ("down arrow" on the phase line). See phase line at left. So the equilibrium at $v = b$ is unstable, and the equilibrium at $v = -b$ is stable.

Our object is *falling* and so its initial velocity must be negative. Hence the phase line says that $v = -b$ is the equilibrium velocity that our object will approach. This is in complete agreement with part (b).

5. [14 points]
Solve

$$(\sec^2 y)y' + (\sec^2 x)\tan y = e^{-\tan x}.$$

Solution. The equation is *not* separable and is definitely *not* linear (due to the $\sec^2 y$). So we look for a substitution to make. The linear, Bernoulli and "y/x" substitutions do not seem to apply. But we know that the derivative of tan is $\sec^2$, and so we try

$$\begin{aligned} v &= \tan y \\ v' &= (\sec^2 y)y' \qquad \text{(remember the } y' \text{ factor, by the chain rule!)} \end{aligned}$$

by the chain rule. So the differential equation says

$$v' + (\sec^2 x)v = e^{-\tan x}.$$

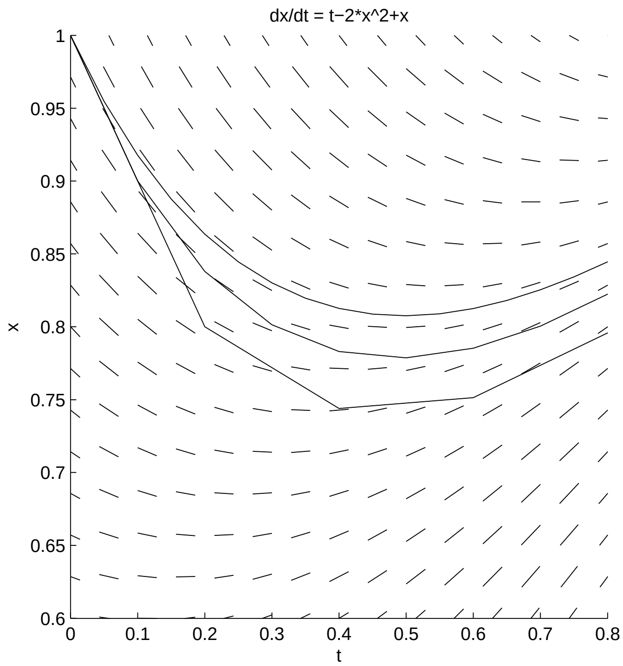
This is first order linear, with integrating factor

$$e^{\int \sec^2 x\, dx} = e^{\tan x}.$$

Multiplying through by this integrating factor yields

$$\begin{aligned} e^{\tan x}v' + (\sec^2 x)e^{\tan x}v &= 1 \\ (e^{\tan x}v)' &= 1 \\ e^{\tan x}v &= x + C \\ v &= e^{-\tan x}(x + C) \\ \tan y &= e^{-\tan x}(x + C) \end{aligned}$$

If desired, one can write this as $y = \arctan(e^{-\tan x}(x + C))$.



6. [15=9+2+4 points]

(a) "Halving the step size halves the error, in Euler's method." Illustrate the meaning of this principle with a suitable sketch above (take $(t_0, x_0) = (0, 1)$).

(b) State the Euler update formula. **Not the Improved Euler update!**

Solution. $\boxed{x_{i+1} = x_i + h \cdot f(t_i, x_i)}$

(c) Consider $\frac{dx}{dt} = \sin(x - t^2)$, with $(t_0, x_0) = (0, 0)$ and $h = 0.1$. Evaluate

$t_1 = t_0 + h = 0.1$

$x_1 = x_0 + h \cdot f(t_0, x_0) = 0 + (0.1) \cdot \sin(0 - 0^2) = 0$

$t_2 = t_1 + h = 0.2$

$x_2 = x_1 + h \cdot f(t_1, x_1) = 0 + (0.1) \cdot \sin(0 - (0.1)^2) = (0.1)\sin(-0.01) \approx -(0.001)$

since $\sin(z) \approx z$ when z is small.

7. [6=2+1+1+2 points] *No explanations are required, on this problem.*

(a) State the definitions of

$$\cosh(x) = \boxed{\tfrac{1}{2}(e^x + e^{-x})}$$
$$\sinh(x) = \boxed{\tfrac{1}{2}(e^x - e^{-x})}$$

Note. It is easy to check that $\cosh' = \sinh$ and $\sinh' = \cosh$.

(b) State the general solution of $y'' - 16y = 0$, in terms of cosh and sinh:

$$y = \boxed{A\cosh(4x) + B\sinh(4x)}$$

(c) State the general solution of $y'' + 16y = 0$, in terms of cos and sin:

$$y = \boxed{A\cos(4x) + B\sin(4x)}$$

Advice. You want to get to the point of being able to write down such solutions without thinking. It helps to practice, by checking the solution:

$$\begin{aligned} y &= A\cos(4x) + B\sin(4x) \\ y' &= -4A\sin(4x) + 4B\cos(4x) \\ y'' &= -16A\cos(4x) - 16B\sin(4x) \end{aligned}$$

so that indeed $y'' + 16y$ does equal 0.

(d) Solve $\frac{dw}{dx} = -2w$ with $w(1) = -1$. Sketch the solution.

Solution. We know the general solution is $\boxed{w = Ce^{-2x}}$. The initial condition says $x = 1$ and $w = -1$, and so $-1 = Ce^{-2}$, or $C = -e^2$. Hence

$$w = -e^2 e^{-2x}.$$

The graph passes through the initial point $(1, -1)$, of course, and decays to zero as $x \to \infty$: