UNDETERMINED COEFFICIENTS — SUMMARY SHEET

Goal: to solve $\boxed{Ly = f}$ where L is a constant coefficient linear differential operator and f has the form

$$\begin{aligned} f(x) &= P_m(x)e^{rx}\cos(kx) \\ \text{or} \qquad &= P_m(x)e^{rx}\sin(kx) \\ \text{or} \qquad &= P_m(x)e^{rx} \end{aligned}$$

where P_m is some polynomial of degree $m \geq 0$, and $r \in \mathbb{R}$ and $k > 0$ are constants.

Method: **find the complementary solution** $y_c(x)$ (the solution of $Ly = 0$). Then...

Rule 1

To find a particular solution $y_p(x)$ of $Ly = f$, try guessing

$$\begin{aligned} y_p(x) &= [A_0 + A_1 x + \cdots + A_m x^m]e^{rx}\cos(kx) \\ &\quad + [B_0 + B_1 x + \cdots + B_m x^m]e^{rx}\sin(kx) \end{aligned}$$

or just $\qquad y_p(x) = [A_0 + A_1 x + \cdots + A_m x^m]e^{rx}$ if there are no sin or cos terms in $f(x)$.

If **no term** in the guess for y_p **duplicates** a term in the complementary solution y_c, then substitute the guess into $Ly = f$ and determine the coefficients $A_0, A_1, \ldots, A_m$ and $B_0, B_1, \ldots, B_m$. Then $y = y_c + y_p$ is the general solution of $Ly = f$.

Rule 2

But if some terms *are* duplicated, change the guess to

$$\begin{aligned} y_p(x) &= x^s[A_0 + A_1 x + \cdots + A_m x^m]e^{rx}\cos(kx) \\ &\quad + x^s[B_0 + B_1 x + \cdots + B_m x^m]e^{rx}\sin(kx) \end{aligned}$$

or just $\qquad y_p(x) = x^s[A_0 + A_1 x + \cdots + A_m x^m]e^{rx}$ if there are no sin or cos terms in $f(x)$,

where s is the smallest exponent that succeeds in eliminating duplication with y_c. That is, keep multiplying through your original guess by powers of x until you have eliminated all duplication.

Then substitute into $Ly = f$ and determine the coefficients $A_0, A_1, \ldots, A_m$ and $B_0, B_1, \ldots, B_m$. Then $y = y_c + y_p$ is the general solution of $Ly = f$.

Note. Figure 3.5.1 in the textbook summarizes the most common guesses for y_p.

What if f consists of more than one term? In that case, find y_p for each term separately, then add the y_p's.

For example, if $f(x) = x^2\sin(x) + e^{4x}$, then we should use Rules 1 and 2 to find a particular solution $y_{p,1}(x)$ corresponding to $f_1(x) = x^2\sin(x)$, and then similarly find a particular solution $y_{p,2}(x)$ corresponding to $f_2(x) = e^{4x}$. Then $y_p = y_{p,1} + y_{p,2}$ will be a particular solution of our original problem.

(This is just an application of the Superposition Principle for nonhomogeneous linear equations.)

Example 1. $y'' + 2y = 4e^{3x}$

Example 2. $y'' - 3y' + 2y = x^2 + x - 5$

Example 3. $2y'' - y' + y = 3\sin(2x)$

Example 4. $y'' + 9y = \cos(3x)$

Example 5. $y'' + 6y' + 16y = e^{-3x}\sin(\sqrt{7}x)$

The annihilator explanation for Rule 2 (duplication).

[For more information, see the *Annihilators* handout on the class website.]