MATH 442 — DAY 35 NOTES

Material: Poisson's equation in 3-dimensions.

- Suppose we want to solve the **Poisson equation**

$$\Delta u(\vec{x}) = -4\pi f(\vec{x}), \quad \vec{x} \in \mathbb{R}^3,$$

with $u(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$. Assume $f(\vec{x}) = 0$ for all large $|\vec{x}|$. (Think of f as describing the mass density of a planet — we'll justify that interpretation later.)

- First, we take $k = 1$ and solve the 3-dimensional diffusion equation

$$v_t = \Delta v, \quad \vec{x} \in \mathbb{R}^3, \qquad \text{with initial value} \quad v(\vec{x}, 0) = 4\pi f(\vec{x}),$$

by writing

$$v(\vec{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x - x^*, t) S(y - y^*, t) S(z - z^*, t) 4\pi f(x^*, y^*, z^*) \, dx^* dy^* dz^*$$

where S is the source function for the diffusion equation in one dimension. (See Exercise 2.4.19 for the justification.)

- We claim the desired solution of Poisson's equation is

$$u(\vec{x}) = \int_0^{\infty} v(\vec{x}, t) \, dt. \tag{1}$$

Proof. The formula for u gives

$$\begin{aligned}
\Delta u(\vec{x}) &= \int_0^{\infty} \Delta v(\vec{x}, t) \, dt \\
&= \int_0^{\infty} v_t(\vec{x}, t) \, dt \\
&= v(\vec{x}, \infty) - v(\vec{x}, 0) \\
&= 0 - 4\pi f(\vec{x}) \qquad \text{since } v(\vec{x}, t) \to 0 \text{ as } t \to \infty.
\end{aligned}$$

And $u(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$ because $S(x - x^*, t) \to 0$ as $x \to \pm\infty$, and so on. □

- Next we evaluate our formula for u: by substituting the definition of v into the formula for u, we find

$$u(\vec{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} S(x - x^*, t) S(y - y^*, t) S(z - z^*, t) \, dt \, 4\pi f(x^*, y^*, z^*) \, dx^* dy^* dz^*.$$

We can evaluate the t-integral:

$$\int_0^\infty S(x-x^*,t)S(y-y^*,t)S(z-z^*,t)\,dt$$

$$= \int_0^\infty \frac{1}{(4\pi t)^{3/2}} e^{-|\vec{x}-\vec{x}^*|^2/4t}\,dt \qquad \text{using the definition of } S$$

$$= \int_0^\infty \frac{1}{(4\pi s)^{3/2}} e^{-1/4s}\,ds \cdot \frac{1}{|\vec{x}-\vec{x}^*|} \qquad \text{where } t = s|\vec{x}-\vec{x}^*|^2$$

$$= \frac{1}{2\pi^{3/2}} \int_0^\infty e^{-r^2}\,dr \cdot \frac{1}{|\vec{x}-\vec{x}^*|} \qquad \text{where } s = 1/4r^2$$

$$= \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}^*|}.$$

- Hence we conclude

$$u(\vec{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|\vec{x}-\vec{x}^*|} f(x^*,y^*,z^*)\,dx^*dy^*dz^*$$

is the solution of **Poisson's equation**

$$\Delta u(\vec{x}) = -4\pi f(\vec{x}), \quad \vec{x} \in \mathbb{R}^3,$$

in 3-dimensions, with $u(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$.

Aside. In 1 or 2 dimensions, the analogous derivation does not work, because the integral in (1) is typically infinite in value. But the end result for u still holds, except with $1/|\vec{x}-\vec{x}^*|$ replaced by $-|x-x^*|$ (in dimension 1) or $\log 1/|\vec{x}-\vec{x}^*|$ (in dimension 2). The constant 4π in Poisson's equation also must change, to 2 (in dimension 1) or 2π (in dimension 2).