MATH 442 — HOMEWORK 10

Section 6.3: 1, 2 (and for #2, express your answer in xy-coordinates as well as $r\theta$-coordinates)

Section 6.2: 3, 6

Additional Problem A: Assume that the neutron density $n(\vec{x}, t)$ inside a lump of Uranium-235 obeys the differential equation

$$\frac{\partial n}{\partial t} = k\Delta n + cn$$

for some positive constant c. (Here c represents the rate at which neutrons in Uranium-235 collide with nuclei and hence cause further neutrons to be released by nuclear fission.) Further suppose the lump is a ball, of radius ρ.

Find the "critical radius" ρ_0 such that if $\rho > \rho_0$ then the neutron density n increases exponentially with time.

Hints. First eliminate the lower order term in the PDE. Then assume n is radial (depends only on r and t) and separate variables. Next use a result from Homework 9. Impose the boundary condition $n = 0$ when $r = \rho$, at the surface of the ball. Require that n be finite at $r = 0$.

Additional Problem B: Assume there is a wire running down the z-axis, having mass density 1 per unit length. The gravitational force of attraction exerted by the wire is

$$\vec{F}(x, y, z) = -\int_{-\infty}^{\infty} \frac{(x, y, z) - (0, 0, z^*)}{|(x, y, z) - (0, 0, z^*)|^3}\, dz^*.$$

(Note the integrand has magnitude $1/|(x, y, z) - (0, 0, z^*)|^2$, which is Newton's inverse-square law for the gravitational force.)

(a) Show $\vec{F}(x, y, z) = \vec{F}(x, y, 0)$. *Hint.* Change of variable.

(b) Let $x > 0$. Evaluate $\vec{F}(x, 0, 0) = -\frac{2}{x}\vec{i} = -\frac{2}{|(x,0,0)|^2}(x, 0, 0)$. *Hint.* $\vec{F}$ has three components. Treat them separately.

(c) Deduce that $\vec{F}(x, y, z) = -\frac{2}{|(x,y,0)|^2}(x, y, 0)$. (A graphical explanation will suffice.)

(d) Show that the potential $u(x, y, z) = 2 \log \frac{1}{s}$ (where $s = \sqrt{x^2 + y^2}$) satisfies $\vec{F} = \nabla u$.

Remarks.

1. In physics we would multiply the potential u by -1 and compensate by defining $\vec{F} = -\nabla u$; it doesn't really matter.

2. Part (d) tells us that the two dimensional analogue of the three dimensional gravitational potential $\frac{1}{r}$ is $\log \frac{1}{s}$. These results fit very nicely with our work on harmonic functions, where we found that $1/r$ is harmonic in three dimensions, and $\log(1/s)$ is harmonic in two dimensions.

3. The same calculations apply in electrostatics, where we consider charge density instead of mass density.