MATH 442 — DAILY READING GUIDES

How to read mathematics. Try to understand every sentence and formula on the pages covered in the Reading Guide for each day, omitting the exercises of course. To learn thoroughly, you need to check every claim for yourself — so keep a pad of paper beside you, and a pen in your hand. A simple example: near the top of page 7, check that Strauss has used the chain rule correctly in showing $u_x = au_{x'} + bu_{y'}$. By doing hundreds of such calculations as you read the book, you will learn the material far better than if you just read the words.

As you work through the material, make a list of the concepts and statements you don't understand. Bring up these questions in class.

Math 442 — Day 1 Reading

First day of class. No preparation required.

Math 442 — Day 2 Reading

Material: Section 1.2.

Review of Chain Rule. Get a calculus book and review the multivariable chain rule. Recall the chain rule works "from the outside in", by differentiating first the outside function and then the inside one.

Use the chain rule to calculate the following expressions, which we will often use:

$$\frac{\partial}{\partial x} f(g(x,y)) = (Df)(g(x,y))\frac{\partial g}{\partial x} = f'(g(x,y))\frac{\partial g}{\partial x}$$

where f is a function of one variable and its derivative is written Df or f'; and

$$\frac{\partial}{\partial x} F(g(x,y),h(x,y)) = (D_1F)(g(x,y),h(x,y))\frac{\partial g}{\partial x} + (D_2F)(g(x,y),h(x,y))\frac{\partial h}{\partial x}$$

where F is a function of two variables and its partial derivatives are written $D_1 f$ and $D_2 f$. (For example, if $F(s,t) = s^2 \sin(t)$ then $(D_1F)(s,t) = 2s\sin(t)$.) The notation D_1F and D_2F relieves us of the burden of keeping track of the names of the two variables inside F, when using the chain rule.

<u>Page 6</u>

- The meaning of directional derivative is recalled on page 387. Here we have $V = (a,b)$ and

  $$0 = au_x + bu_y = V \cdot \nabla u = (\text{directional derivative of } u \text{ in direction } V).$$

- "Orthogonal" means perpendicular (dot product equals zero).

<u>Page 7</u>

- The letters x' and y' denote new variables. (The $'$ does not mean a derivative.)

<u>Page 8</u>

- Check the formula after (6) (the one starting with $\frac{d}{dx}u(x,Ce^x)$), by using the Chain Rule and the notation D_1u, D_2u, and so on. This formula is the key to the whole example.

<u>Page 9</u>

- Skip Example 3.
- Check what the "In summary" paragraph is saying, by referring it back to Examples 1 and 2.

Math 442 — Day 3 Reading

General advice. Read actively, meaning: keep a pad of paper beside you and a pen in your hand, and write out the calculations for yourself, to check them. As you work, note down the statements that puzzle you. Ask about them next time.

Material: Section 1.3. Concentrate on pages 10,11,12, and Example 4 (which we will study further in class).

Page 10

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Page 11 (through the end of Example 1)

- The differentiation with respect to b uses an important part of the Fundamental Theorem of Calculus, a part that is often overlooked:

$$\frac{d}{db}\int_a^b f(x)\,dx = f(b). \tag{1}$$

  Please look back at your calculus book and make sure you understand what this formula is saying. *Notes.* Here, x is just a dummy variable of integration, a is a fixed number, and the upper limit b of integration is the variable. More precisely, if we write $F(b) = \int_a^b f(x)\,dx$ for the value of the integral from a up to b, then formula (1) says $F'(b) = f(b)$.
- The triangle in Figure 3 is useful: the slope of the string is u_x, and since the tension T acts along the hypotenuse, the horizontal component of tension is $T\cos\theta = T \cdot \frac{1}{1+u_x^2}$, while the vertical ("transverse") component of tension is $T\sin\theta = T \cdot \frac{u_x}{1+u_x^2}$.

Page 12

- For the second equation (transverse component of Newton' Law), note that $\rho\,dx$ represents the mass contained in the length dx, and so $\rho u_{tt}\,dx$ represents mass $\times$ acceleration, which we should then integrate from x_0 to x_1 to get the total "ma" for Newton's Law.
- "The second equation, differentiated,..." means to replace x_1 by x and then differentiate the equation with respect to x (noting that x_0 is a constant).
- In (i), ask yourself why the constant r ought to be positive rather than negative, in order to model air resistance.

Pages 13 and 14 — Example 3

- Omit. Just observe how the 2D and 3D wave equations (6) and (7) are generalizations of the 1D wave equation (2).

Pages 14 and 15 — Example 4

- The derivation of the diffusion equation (8) on page 15 is important. It proceeds by conservation of mass of the dye, and uses Fick's law, which claims the flux through an endpoint is proportional to the concentration gradient at that endpoint.

  WARNING: Figure 5 is seriously misleading, for it suggests that the fluid is flowing from left to right, whereas in fact the fluid is motionless and it is the *dye* that *diffuses through* the fluid. And the "flow in" and "flow out" phrases at the top of page 15 should be deleted, since it is not necessarily true that the dye flows in at x_1 and out at x_0.

Here is a correct statement: at the right endpoint x_1 of our interval, Fick's Law says that the *inward* flux of dye is $ku_x(x_1,t)$ for some constant $k > 0$. [Inward flux means "mass per unit time passing in to the interval".] Is Fick's Law sensible? Let us check: suppose $u_x(x_1,t) > 0$, so that the dye concentration is bigger to the right of x_1 than to the left of x_1, at time t. Then the dye will want to diffuse from the right of x_1 towards the left, in order to "even out the concentrations". Therefore the inward flux will be positive, meaning the inward flux $ku_x(x_1,t)$ term in Fick's Law does seem fairly plausible.

Convince yourself that the inward flux at the left end x_0 similarly ought to be $-ku_x(x_0,t)$. Now you understand the equation $\frac{dM}{dt} = ku_x(x_1,t) - ku_x(x_0,t)$ at the top of page 15!

Skim lightly over the 3D diffusion equation.

Pages 15 and 16 — Examples 5 and 6

- Skim lightly over Examples 5 and 6. Heat flow is directly analogous to the diffusion of dye in fluid, with temperature playing the role of the dye concentration. e.g. Fourier's law says that heat flows from high temperature regions to low temperature regions, with the heat flux being proportional to the temperature gradient. This is analogous to Fick's law. And Example 5 derives the heat equation from conservation of heat energy, just like the dye diffusion equation is derived in Example 4 from conservation of mass.

Pages 17 and 18

- Example 7 is optional. It probably won't make sense anyway, unless you have already studied quantum mechanics.

Math 442 — Day 4 Reading

Material: Section 1.4. Read pages 19-21, and the first half of page 22.

The first couple of paragraphs describe INITIAL CONDITIONS. Then most of the section discusses BOUNDARY CONDITIONS.

Try to understand the *physical meaning* of the boundary conditions, for the vibrating string, diffusion and heat problems. Draw pictures to illustrate.

Skip the section on sound (acoustics), which requires considerable physical knowledge.

Page 21

- For the vibrating string with a *free* end (meaning the endpoint is free to slide up and down a frictionless transverse guide-rail), why must $u_x = 0$ at the endpoint? (Consider the tension force at that endpoint...)

Math 442 — Day 5 Reading

Material: Section 1.5, and synthesis of Sections 1.1–1.4.

Page 25

- Try the following problem, to test your understanding of "stability". Consider the ordinary differential equation $\frac{dx}{dt} = -2x$, with initial data $x(0) = a$. Consider two initial values, a_1 and a_2, and write $x_1(t)$ and $x_2(t)$ for the corresponding solutions. Find these solutions, and then show that for any fixed t-value,

  $$x_1(t) - x_2(t) \to 0 \qquad \text{as } a_1 \to a_2.$$

  Intuitively, this means that if the initial values are close together, then the solutions themselves are close together at time t. That is what stability means for an ODE. (For a PDE, we would need to consider two solution having different initial values and different boundary values.)

Page 26

- This page is a little vague, but try to understand the spirit of it. *Aside.* To learn about metrics and norms of functions, I recommend Math 432 or 447.
- The third example, on matrices, is optional. So is the fourth example, on Laplace's equation.

Synthesize Sections 1.1–1.4

For each section, write a short paragraph summarizing the main points. Ask yourself a question, based upon the material in each section. Bring that question to class.

Math 442 — Day 6 Reading

Material: Section 2.1.

Page 32

- Factoring the wave equation: check that the lefthand equality in equation (2) is correct.

Page 33

- Skip the first half of the page (solving the factored wave equation).
- Check all the calculations in the second half of the page (the method of *characteristic coordinates*).

Page 34 until halfway down page 35 — D'Alembert's formula

- This derivation was covered in Math 285. We will review it in class.

Pages 35 and 36 — Examples 1 and 2

- In Example 2, the formula that Strauss finds for u is not important. The important thing is Figure 2 on page 36: can you use D'Alembert's formula (in the box on page 35) to explain why each snapshot of the wave in Figure 2 looks the way it does?

Math 442 — Day 7 Reading

Material: Section 2.2.

Pages 37 and 38 — Causality

- Ignore the first paragraph of Section 2.2; it is hard to understand.
- To justify the "domain of dependence" in Figure 2 on page 38, look at the D'Alembert formula (in the box on page 35) and notice the value of $u(x,t)$ depends on: the value of the initial displacement ϕ at $x - ct$ and at $x + ct$, and the values of the initial velocity ψ between $x - ct$ and $x + ct$. Hence the domain (or interval) of dependence of u at the point (x,t) is the interval $[x - ct, x + ct]$.
- Can you now justify the "domain of influence" in Figure 1 on page 38, by using Figure 2?

Pages 38 and 39 — Energy

- No reading — we will cover this material in class.
- Ignore the last paragraphs of the section, after Example 1, because they are obscurely written. Here is what Strauss means:

  The traveling wave formula $u(x,t) = f(x + ct) + g(x - ct)$ in Section 2.1 shows that in one dimension, u can be broken into a sum of two waves traveling at speed exactly c — never "slower than speed c" as Strauss says can happen. What Strauss means is that these two waves can interact so as to create a lasting disturbance near any given point, *e.g.* exercise 2.1 #5 (*Hammer blow*). Such lasting disturbances can also occur in two dimensions, but not in three dimensions (see Chapter 9). Using picturesque language, we say signals are "sharp" in three dimensions, but not in one or two.

Math 442 — Day 8 Reading

Material: Parts of Section 2.3

Pages 41 and 42 — Maximum Principle

- Do not read the Proof of the Maximum Principle. I will lecture on it in class.
- But do ask yourself: what does the Maximum Principle mean physically?
- Note the Maximum Principle is true for any boundary and initial conditions.

Math 442 — Day 9 Reading

Material: Parts of Section 2.3.

Pages 42 and 43 — Uniqueness

- The technique for proving uniqueness of the solution is: suppose there are two solutions, u_1 and u_2, and let $w = u_1 - u_2$. Then figure out what differential equation and boundary conditions and initial conditions are satisfied by w. Then use either a maximum principle method or else an energy method, to deduce $w \equiv 0$, so that u_1 and u_2 are the same.
- Read the maximum principle method for proving uniqueness.
- In the discussion of the "energy method" in Strauss, you can ignore everything up until formula (4). That formula expresses the dissipation of the heat energy: it says $E(t) \leq E(0)$. [The work that leads up to formula (4) is Strauss' way of showing how to figure out what "energy" one ought to use; but in class we already guessed $E(t) = \frac{1}{2} \int u^2 \, dx$, and showed that it is dissipated.]
- The three lines after formula (4) explain how to use the energy method to prove uniqueness.

Math 442 — Day 10 Reading

Material: The first half of Section 2.4.

Pages 45 and 46 — Invariance Properties of the Diffusion Equation

- Read and justify properties (a)–(e). For instance, in (a) you fix a number y and let $v(x,t) = u(x-y,t)$. Assume u satisfies the diffusion equation, and show v does also.
- Similarly in (e) you fix a number $a > 0$ and let $v(x,t) = u(\sqrt{a}x, at)$ and show that if u satisfies the diffusion equation then so does v.

Page 46 — Step 1

- Step 1 says the following. Fix $a > 0$, and define $\tilde{Q}(x,t) = Q(\sqrt{a}x, at)$. Then $\tilde{Q}$ satisfies the diffusion equation by property (e), and you can check that $\tilde{Q}(x,0) = 1$ if $x > 0$ and $\tilde{Q}(x,0) = 0$ if $x < 0$. Since $\tilde{Q}$ and Q satisfy the same PDE and the same initial condition, we "expect" $Q = \tilde{Q}$, that is

$$Q(x,t) = Q(\sqrt{a}x, at) \qquad \text{for all } x \in \mathbb{R}, \quad t > 0, \quad a > 0.$$

  Substituting $a = 1/4kt$ now gives $Q(x,t) = Q(p, 1/4k)$, so that Q is just a function of p. That is what equation (4) says!

  This kind of "scaling" or "self-similarity" idea is very powerful in PDE.

Page 46 — Steps 2 and 3

- These steps will be covered in class.

Math 442 — Day 11 Reading

Material: The second half of Section 2.4.

Pages 47 and 48 — Step 4

- Strauss defines $u(x,t)$ in formula (6), and then shows u satisfies the PDE.
- How does Strauss show $u(x,t)$ satisfies the initial condition $u(x,0) = \phi(x)$? Why can't he do it just by putting $t = 0$ into the definition of u?

Pages 48 and 49 — Discussion of the Source Function

- Try to justify the following statement, which Strauss basically makes on page 49.
  "Let y_0 be fixed. Then the function $S(x-y_0,t)$ gives the chemical concentration at position x and time t that results from a unit of chemical being released at position y_0 at time 0".

Page 50 — Examples 1 and 2 are optional

- Example 2 is not as bad as it looks. Note the variable p that is introduced in order to evaluate the integral is different from the "p" that was used earlier in the section.

Math 442 — Day 12 Reading

Material: Section 2.5, and general re-cap on Chapter 2.

Pages 52–54 — Section 2.5

- Write a sentence explaining properties (i), (iii), (v), (vi), for Waves and for Diffusions, by referring back to things we have done in class or formulas we know.
- Property (iii): We have not examined continuous dependence on initial data for the wave equation, for $t > 0$ or $t < 0$. But this "continuous dependence" follows easily from the D'Alembert formula, which expresses $u(x, t)$ explicitly in terms of the initial data.

Pages 32–54 — Chapter 2

- Look back over Chapter 2. For each section, write down a topic or formula or method that you would like to spend more time on, or would like to ask a question about.

Math 442 — Day 13 Reading

Material: Section 3.1.

Page 55

- Strauss makes a false statement about the physical interpretation of the Dirichlet boundary condition in (1). What is the false statement, and how should it be corrected?

Page 56

- We will show in class that $u(x,t)$ in formula (4) is an odd function, meaning $u(-x,t) = -u(x,t)$. You are welcome to try showing this before class.

Page 57

- Read up until Example 1.
- Example 1 is optional. Recall Erf was defined in Section 2.4. Typo in formula (8): the second occurrence of "$x \geq 0$" should be "$x \leq 0$".

Math 442 — Day 14 Reading

Material: Section 3.2 (pages 59 and 60 only).

Page 59

- Recall that an odd function f has the property $f(-x) = -f(x)$ for all real numbers x. Hence when you put $x = 0$ into the formula for $v(x, t)$ near the bottom of page 59 (before formula (2)), you get $v(0, t) = 0$. This is the desired Dirichlet boundary condition.

Page 60

- The main intuitive point is that when a left-moving wave hits the Dirichlet boundary condition at $x = 0$, it gets *reflected* so as to become right-moving. But it gets reflected *upside down* due to the minus sign in front of $\phi(ct - x)$ in formula (3).

Page 61–63 — The Finite Interval (**optional material**)

- Do *not* read this material carefully. The most interesting ideas are in Figure 3 (showing how you can extend the initial displacement) and Figure 4 (showing how the edge of a travelling wave will get reflected repeatedly as it hits first the right endpoint then the left endpoint).
- A better way to get a feeling for reflections of traveling waves is to play around with Iode or other wave equation software and see the reflections happening.

Math 442 — Day 15 Reading

Test 1 is on Day 15, and so there is **no** reading assignment.

Math 442 — Day 16 Reading

Material: Test feedback, and Section 3.3 Diffusion with a Source.

Pages 65–67 — Diffusion with a Source

- The analogy between the PDE (1) and the ODE (3) is important. Read this page down through formula (4).
- I will lecture on the proof of formula (2). You do not need to read it ahead of time.

Math 442 — Day 17 Reading

Material: Section 3.4 Waves with a Source.

Page 69 — Waves with a source

- Notice formula (3) consists of the D'Alembert formula plus a term involving the external force f.
- Check that you understand the limits of integration in the iterated integral that comes after (3).
- I will lecture on the proof of formula (3). For preparation, just read the first page of Section 3.4.

Math 442 — Day 18 Reading

Example — Temperatures underground

- We will finish working through the following example from the last class. Imagine the x-axis points down into the ground, with $x = 0$ being at ground level. The heat equation $u_t = ku_{xx}$ applies on the half-line $x > 0$, underground.
- We assume a nonhomogeneous boundary condition $v(0,t) = h(t)$, where the soil temperature $h(t)$ at ground level will fluctuate slowly during the year due to the changing seasons. (Day-to-day fluctuations in soil temperature tend to be small.)
- In class, we will solve for $v(x,t)$. No preparation required for this class.

MATH 442 — DAY 19 READING

Material: Section 4.1 Separation of Variables, The Dirichlet Condition

<u>Pages 82–84</u>

- This material should be familiar to you from Math 285.
- Check directly that formula (9) really is a solution of the wave equation.
- Check directly that putting $t = 0$ into (9) gives the initial conditions (10) and (11).
- In Figure 1, can you identify the functions $\sin(\pi x/l), \sin(2\pi x/l), \sin(3\pi x/l)$?

<u>Page 85</u>

- Formula (12) uses that the wavespeed for a vibrating string is $c = \sqrt{T/\rho}$.
- Check directly that formula (17) really is a solution of the diffusion equation.
- Check directly that putting $t = 0$ into (17) gives the initial condition (18).

Math 442 — Day 20 Reading

Material: Section 4.1, continued, and Section 4.2 Separation of Variables, The Neumann Condition

Page 86

- The analogy between eigenvalues of $-\frac{d^2}{dx^2}$ and eigenvalues of a matrix is nice to observe.
- The bottom part of the page shows that all eigenvalues of this differential operator are positive, under Dirichlet boundary conditions. We also do this in class by the "multiply by X and then integrate" method.
- Ignore the material on complex eigenvalues.

Aside. Iode uses the solution formulas developed in Sections 4.1 and 4.2, when it solves the wave and heat equations on the interval $0 < x < l$.

Section 4.2 — The Neumann Condition

- Verify directly that formula (5) satisfies the diffusion equation and satisfies the Neumann boundary conditions $u_x(0,t) = u_x(l,t) = 0$.
- Put $t = 0$ into (5) to obtain (6).
- Verify directly that formula (7) satisfies the wave equation and satisfies the Neumann boundary conditions $u_x(0,t) = u_x(l,t) = 0$.
- Put $t = 0$ into (7) to obtain (8) and (9).

Section 4.3 — The Robin Condition

- *Optional material.* The Robin boundary condition is physically important, but the mathematical analysis of it becomes quite intricate. Still, the section is worth skimming if you need it in future.

Math 442 — Day 21 Reading

Material: Supplement to Chapter 4 — **Orthogonality of Eigenfunctions**

This reading assignment develops some orthogonality properties of eigenfunctions, to be used in Chapter 5 for Fourier Series. You need to read parts 1–4 below, and bring your questions on this material to class.

1. Definition of Orthogonality

We say functions $f(x)$ and $g(x)$ are *orthogonal* on $a < x < b$ if $\boxed{\int_a^b f(x)g(x)\,dx = 0}$. [Motivation: Let's approximate the integral with a Riemann sum, as follows. Take a large integer N, put $h = (b-a)/N$ and partition the interval $a < x < b$ by defining $x_1 = a+h, x_2 = a+2h, \ldots, x_N = a + Nh = b$. Then

$$\begin{aligned}\int_a^b f(x)g(x)\,dx &\approx f(x_1)g(x_1)h + \cdots + f(x_N)g(x_N)h \\ &= (u_N \cdot v_N)h\end{aligned}$$

where $u_N = (f(x_1), \ldots, f(x_N))$ and $v_N = (g(x_1), \ldots, g(x_N))$ are vectors containing the values of f and g. The vectors u_N and v_N are said to be orthogonal (or perpendicular) if their dot product equals zero ($u_N \cdot v_N = 0$), and so when we let $N \to \infty$ in the above formula it makes sense to say the functions f and g are orthogonal when the integral $\int_a^b f(x)g(x)\,dx$ equals zero.]

Example. $\sin x$ and $\cos x$ are orthogonal on $-\pi < x < \pi$, since $\int_{-\pi}^{\pi} \sin x \cos x\,dx = \frac{1}{2}\sin^2 x\big|_{-\pi}^{\pi} = 0$.

2. Integration Lemma

Suppose functions $X_n(x)$ and $X_m(x)$ satisfy the differential equations

$$\begin{aligned} X_n'' + \lambda_n X_n &= 0, \qquad && a < x < b, \\ X_m'' + \lambda_m X_m &= 0, && a < x < b, \end{aligned}$$

for some numbers λ_n, λ_m. Then

$$(\lambda_n - \lambda_m)\int_a^b X_n(x)X_m(x)\,dx = [X_n(x)X_m'(x) - X_n'(x)X_m(x)]_a^b.$$

Proof.

$$\begin{aligned} \text{LHS} &= \int_a^b [(\lambda_n X_n)X_m - X_n(\lambda_m X_m)]\,dx && \text{by taking the } \lambda\text{'s inside the integral} \\ &= \int_a^b [-X_n'' X_m + X_n X_m'']\,dx && \text{since } \lambda_n X_n = -X_n'' \text{ and } \lambda_m X_m = -X_m'' \\ &= \int_a^b [X_n X_m' - X_n' X_m]'\,dx && \text{as you can check by differentiating!} \\ &= \text{RHS} && \text{by the Fundamental Theorem of Calculus.} \end{aligned}$$

□

3. Boundary Conditions

Consider a function $X(x)$ for $a < x < b$. We consider five boundary condition ("BC") types:

$$\begin{aligned} \text{Dirichlet BC} \qquad & X(a) = X(b) = 0 \\ \text{Neumann BC} \qquad & X'(a) = X'(b) = 0 \\ \text{Mixed BC} \qquad & X(a) = X'(b) = 0 \\ \text{Mixed BC} \qquad & X'(a) = X(b) = 0 \\ \text{Periodic BC} \qquad & X(a) = X(b), \quad X'(a) = X'(b) \end{aligned}$$

(The two varieties of Mixed BC are very similar, differing only as to which endpoint has $X = 0$ and which has $X' = 0$.)

4. **Orthogonality of Eigenfunctions Theorem**

Suppose $X_n'' + \lambda_n X_n = 0$ and $X_m'' + \lambda_m X_m = 0$ on $a < x < b$, and that X_n and X_m both satisfy the same type of BC.

If $\lambda_n \neq \lambda_m$ then X_n and X_m are orthogonal, meaning

$$\int_a^b X_n(x) X_m(x)\, dx = 0.$$

Proof. By the Integration Lemma, we have

$$\begin{aligned} &\int_a^b X_n(x) X_m(x)\, dx \\ &\qquad = \frac{1}{\lambda_n - \lambda_m} [X_n(x) X_m'(x) - X_n'(x) X_m(x)]_a^b \\ &\qquad = 0 \quad \text{under Dirichlet BCs, because } X_n(a) = X_n(b) = 0 \text{ and } X_m(a) = X_m(b) = 0 \\ &\qquad = 0 \quad \text{under Neumann BCs, because } X_n'(a) = X_n'(b) = 0 \text{ and } X_m'(a) = X_m'(b) = 0 \\ &\qquad = 0 \quad \text{under Mixed BCs, for similar reasons.} \end{aligned}$$

For periodic BCs, we use that $X_n(b) = X_n(a)$ and $X_m'(b) = X_m'(a)$ and so on, to see

$$\begin{aligned} [X_n(x) X_m'(x) - X_n'(x) X_m(x)]_a^b = {} & [X_n(b) X_m'(b) - X_n'(b) X_m(b)] \\ & - [X_n(a) X_m'(a) - X_n'(a) X_m(a)] = 0. \end{aligned}$$

□

5. Example

Show that $\sin x$ and $\sin 2x$ are orthogonal for $0 < x < \pi$. *Solution.* We could just show $\int_0^\pi \sin(x)\sin(2x)\, dx = 0$ by using trigonometric identities to evaluate the integral. But it is easier to notice that both $X_1(x) = \sin(x)$, with $\lambda_1 = 1^2$, and $X_2(x) = \sin(2x)$, with $\lambda_2 = 2^2$, are eigenfunctions for

$$\begin{aligned} X'' + \lambda X = 0, \qquad & 0 < x < \pi, \\ X(0) = X(\pi) = 0 \qquad & \text{(Dirichlet BC)}. \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, the Orthogonality Theorem implies $\int_0^\pi X_1 X_2 \, dx = 0$, so that $\sin(x)$ and $\sin(2x)$ are orthogonal for $0 < x < \pi$.

6. Orthogonality of sines and cosines for $-\pi < x < \pi$

The following formulas will be essential for our study of Fourier series, in Chapter 5. Let $n, m \geq 1$ be integers. Then:

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = 0 \quad \text{if } n \neq m \tag{2}$$

$$= \pi \quad \text{if } n = m, \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = 0 \quad \text{if } n \neq m \tag{4}$$

$$= \pi \quad \text{if } n = m, \tag{5}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\, dx = 0 \quad \text{if } n \neq m \tag{6}$$

$$= 0 \quad \text{if } n = m. \tag{7}$$

Proof. For formula (2), we let $X_n(x) = \cos(nx)$ and $X_m(x) = \cos(mx)$, so that X_n and X_m are eigenfunctions of $X'' + \lambda X = 0$ on $-\pi < x < \pi$, with eigenvalues $\lambda_n = n^2$ and $\lambda_m = m^2$ respectively. Also, X_n and X_m satisfy **periodic** boundary conditions (since they are both 2π-periodic). If $n \neq m$ then $\lambda_n \neq \lambda_m$, and so the Orthogonality Theorem implies X_n and X_m are orthogonal, which is equation (2). Equations (4) and (6) are proved similarly. [Exercise.]

To get equation (3), we substitute $n = m$ and evaluate the integral explicitly as

$$\int_{-\pi}^{\pi} \cos^2(mx)\, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos(2mx))\, dx = \pi.$$

Equation (5) proceeds similarly. [Exercise.] And finally, equation (7) is true since we can substitute $n = m$ and evaluate the integral explicitly as

$$\int_{-\pi}^{\pi} \cos(mx)\sin(mx)\, dx = \frac{1}{2m}\sin^2(mx)\Big|_{-\pi}^{\pi} = 0.$$

Remark. Here we have used the Orthogonality Theorem to evaluate integrals (2), (4) and (6). They can also be evaluated using trigonometric identities. But "orthogonality of eigenfunctions" is the best way to think about these results, which we will need in Chapter 5 for Fourier Series. ∎

MATH 442 — DAY 22 READING

Material: Section 5.1

Pages 101–102 — Fourier sine series

- The orthogonality of the sine functions in formula (2) is known to us already, from our work in the Orthogonality Supplement (although the proof on page 102 by trigonometric identities is also nice).
- The derivation of (4) is **really** important. Try hard to understand it, and bring your questions to class.
- *Exercise 1.* Solve the diffusion equation on $0 < x < l$ with Dirichlet boundary conditions and initial condition $u(x, 0) = \phi(x)$. *Solution:* By separation of variables in Chapter 4 we know the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right).$$

  Putting $t = 0$ gives $\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$, so that $A_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx$ for $n \geq 1$, by formula (4).
- *Exercise 2.* Write down a corresponding solution for the wave equation on $0 < x < l$ with Dirichlet boundary conditions and with initial conditions $u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$. (Formula (5) helps.)

Page 103 — Fourier cosine series

- The orthogonality of the cosine functions is known to us already, from our work in the Orthogonality Supplement.
- *Exercise 3.* Solve the diffusion equation on $0 < x < l$ with Neumann boundary conditions and initial condition $u(x, 0) = \phi(x)$. *Solution:*

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos\left(\frac{n\pi x}{l}\right)$$

  where $A_n = \frac{2}{l}\int_0^l \phi(x) \cos\left(\frac{n\pi x}{l}\right)\, dx$ for $n \geq 0$, by formula (7).

Math 442 — Day 23 Reading

Material: Section 5.1 continued

- Finish working through the Day 22 Worksheet. We will briefly review that Worksheet at the beginning of class, before continuing with Section 5.1.

Math 442 — Day 24 Reading

Material: Parts of Section 5.2, and 5.4.

Page 112–113 — Complex form of the full Fourier series

- The "complex" Fourier series formula (12), namely
$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \quad \text{with coefficients} \quad c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x) e^{-in\pi x/l}\, dx,$$
is how mathematicians remember the Fourier series. This formula is shorter and more elegant than the "real" formula in terms of sines and cosines:
$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}).$$
- Show that the two formulas are related by
$$A_0 = 2c_0$$
$$A_n = c_n + c_{-n}$$
$$B_n = ic_n - ic_{-n}$$

Pages 121–122 — Three notions of convergence

I will lecture on this material in class. After class, do the following:

- Re-read the three definitions of convergence carefully.
- Work in detail through Example 1, referring back to the definitions to justify each claim about convergence (namely, that the series converges pointwise, does not converge uniformly, and does converge in the mean-square sense).
- Note that Figure 2 shows *partial sums* $\sum_{n=1}^{N} f_n(x)$ for various values of N. (The graphs do *not* show the individual functions $f_n(x)$.)

Math 442 — Day 25 Reading

Material: Section 5.4 continued

After class, review your notes and textbook carefully, checking every step of the work we did in class.

Ignore Theorem 2 (Uniform convergence) as stated in Strauss page 124. Theorem 2 stated in class is better.

Math 442 — Day 26 Reading

Material: Section 5.4, continued

After class, review your notes and textbook carefully, checking every step of the work in class. In particular:

Pages 125–126

- Work through Example 3 to see what the three convergence theorems tell you, or don't tell you, about this example.
- Use a convergence theorem to show that the Fourier cosine series of $x^2, 0 < x < l$, indeed converges to x^2 at every point $0 \le x \le l$, including in particular at $x = l$. [We used this fact on page 2 of the Compendium, in order to show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

Math 442 — Day 27 Reading

Material: Section 5.5.

Pages 132–33 — Pointwise convergence of Fourier series

- We take $l = \pi$ and assume $f(x)$ is 2π-periodic, with $f'(x)$ being assumed a continuous function.
- These pages develop a simple expression for the partial sum $S_N(x)$ of the Fourier series.
- *Typo:* There is a minus sign missing in the second-to-last equation in the proof of (5). Formula (5) itself is correct.

Pages 136–138 — Gibbs phenomenon

- This material is rather difficult, and is optional.
- The final conclusion, in formula (21), is that
$$S_N\left(\frac{\pi}{N+\frac{1}{2}}\right) \to 0.59\ldots \qquad \text{as } N \to \infty.$$
That is, we are examining the Nth partial sum at a point $x_N = \frac{\pi}{N+\frac{1}{2}}$ that gets closer and closer to the jump point ($x = 0$) as $N \to \infty$. The conclusion is that the Nth partial sum at x_N approaches 0.59 instead of 0.5, even though 0.5 is the value of the function $f(x)$ for all $x > 0$. Thus the overshoot point x_N gets closer and closer to the jump point, but the *magnitude* of the overshoot does not diminish: it remains about 0.09, which is 9% of the magnitude of the jump in the value of the function from $-\frac{1}{2}$ to $\frac{1}{2}$.
- *Aside.* You will have seen the Gibbs phenomenon in your Iode work on Fourier series. The phenomenon is simply inescapable when you use Fourier series to approximate functions having jumps.

Math 442 — Day 28 Reading

Material: Section 5.6.

Section 5.6 answers two questions for the wave and diffusion equations on the interval $0 < x < l$: how does one handle nonhomogeneous boundary conditions, and how does one handle a nonhomogeneous term in the PDE?

Pages 140–143

- We will cover this material in class, with a slightly different method. No reading required.

Pages 143

- "Shifting the data" simply means that by subtracting off a function that satisfies the boundary conditions, we can reduce to the case of homogeneous boundary conditions $h \equiv 0, k \equiv 0$ (at the cost of introducing a nonhomogeneous term into the PDE).

  I generally prefer to "shift the data" before solving a PDE with nonhomogeneous boundary conditions.

MATH 442 — DAY 29 READING

Material: Section 6.1 — Laplace's equation.

If u is a steady state (time independent) solution of the diffusion equation $u_t = k\Delta u$ or the wave equation $u_{tt} = c^2\Delta u$, then u satisfies Laplace's equation $\boxed{\Delta u = 0}$.

<u>Pages 146–147: definitions and examples</u>

- In the "Electrostatics" example, recall from vector calculus that a vector field $\mathbf{E}(x, y, x)$ has zero curl if and only if $\mathbf{E}$ is a gradient field, meaning $\mathbf{E} = -\nabla\phi$ for some real-valued "potential" function $\phi(x, y, z)$.
- For the "Analytic functions" example, take $f(z) = z^2 = (x + iy)^2$ and evaluate the real and imaginary parts of $f(z)$ (which are called u and v, in Strauss). This example will give some examples of harmonic functions.
- Some deep mathematics lies behind the "Brownian motion" example on page 147, and it is not expected that you understand how it works.

<u>Pages 148–149: Maximum Principle for Laplace's equation</u>

- Read both the statement and the proof.

Math 442 — Day 30 Reading

Material: Section 6.1 — Laplace's equation, cont.

Pages 150–151: Invariance in two dimensions

- An equivalent way of stating the *translation* invariance of the Laplacian is: let $v(x, y) = u(x + a, y + b)$ then show
$$(\Delta v)(x, y) = (\Delta u)(x + a, y + b),$$
or in other words
$$v_{xx}(x, y) + v_{yy}(x, y) = u_{xx}(x + a, y + b) + u_{yy}(x + a, y + b).$$
This is easy to do by the chain rule.
- An equivalent way of stating the *rotation* invariance of the Laplacian is: let
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
define the rotation of $\begin{pmatrix} x \\ y \end{pmatrix}$ through angle α, and put $v(x, y) = u(x', y')$; then show
$$(\Delta v)(x, y) = (\Delta u)(x', y').$$
This is not hard to do by the chain rule.
- Laplacian in polar coordinates: we are not going to prove formula (5) in class. You should do it yourself, at least once in your life. I recommend using the chain rule to find
$$u_r = u_x x_r + u_y y_r = \cos\theta u_x + \sin\theta u_y$$
and then find u_{rr} and $u_{\theta\theta}$ similarly. Then substitute into the righthand side of (5). This approach is easier than Strauss' method.

Math 442 — Day 31 Reading

No reading assignment, and no class, because of Test 2.

Math 442 — Day 32 Reading

Material: Section 6.1 continued, and Section 6.3.

Pages 152–153: Invariance in three dimensions

- Laplace's equation is invariant under translations and rotations. For translations, this means that if $u(x, y, z)$ is harmonic, then so is the function $v(x, y, z) = u(x + a, y + b, z + c)$. For rotations, this means that if $u(\mathbf{x})$ is harmonic (where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$), then so is the function $v(\mathbf{x}) = u(B\mathbf{x})$ where B is an orthogonal (rotation) matrix.
- Formula (6) expresses the Laplacian in spherical coordinates. Examine Figure 3 carefully to see which angle is θ and which is ϕ! We have

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta.$$

- We will not verify formula (6) in class. You should do it once in your life. The derivation in Strauss is clever, but you might find it easier to just start with the righthand side of (6) and use the chain rule to express u_ϕ in terms of u_x and u_y and u_z, and so on.
- Definitely work through the last paragraph of page 153 to show that radial harmonic functions in 3 dimensions must have the form

$$u(r) = \frac{\text{const}}{r} + \text{const}.$$

  Notice that is the form of gravitational or electrostatic potentials, decaying like $1/r$.

Math 442 — Day 33 Reading

We will spend the day debriefing on Test 2, and then finishing Section 6.3.

Pages 162–163: Maximum principle

- This paragraph shows that if a harmonic function u attains its maximum at an interior point, then $u \equiv$ constant. Hence for nonconstant harmonic functions, the maximum and minimum are *only* attained at the boundary. (Previously we had proved the maximum and minimum are attained at the boundary, but we did not rule out the possibility of them also being attained inside the domain.)

Math 442 — Day 34 Reading

Material: Section 6.2.

Pages 155–156: Laplace's equation in a rectangle

- Skim Example 1 and identify the main steps. We will build on this example in class.

Math 442 — Day 35 Reading

Material: Section 7.4.

Pages 185–186: Poisson formula in 3-dimensions

- Consider formulas (16) and (17) for a few minutes. (You are not responsible for the proofs.)
- Formula (17) on page 186 is the "traditional" Poisson formula for solving $\Delta u = 0$ in the ball of radius a with $u = h$ on the boundary sphere of radius a. We have already studied the analogous formula in 2-dimensions, formula (13) on page 161.
- Formula (16) on page 185 is the "geometric" version of the Poisson formula. The analogous formula in 2-dimensions is (14) on page 162.
- WARNING: Poisson's formula solves Laplace's equation $\Delta u = 0$ (not Poisson's equation!)

Math 442 — Day 36 Reading

Material: Section 10.3 — Solid vibrations in a ball.

Pages 245–246: Separating out t in 3-dimensions

- Read this background information down, to (5).
- Just like in one dimension (in the Orthogonality Supplement), one can show eigenfunctions corresponding to distinct eigenvalues are automatically orthogonal, under Dirichlet or Neumann BC.

Pages 257–258: Separating the spatial variables in 3-dimensions

- Work through this material line by line, down to equation (3).
- Equation (2) is similar to Bessel's equation on page 252.

In class we will further discuss Bessel functions, and will begin to separate the angular variables.

Math 442 — Day 37 Reading

Material: Section 10.3, continued.

Pages 259–260: Separating the θ and ϕ variables

- To get from (12) to (14), use the chain rule:
$$\frac{d}{d\theta} = \frac{ds}{d\theta}\frac{d}{ds} = -\sin\theta\frac{d}{ds}.$$
- You can think of the new variable $s = \cos\theta$ as equalling the vertical coordinate of a point on the unit sphere (since θ is the latitudinal angle, measured from the north pole).
- To understand the eigenvalue equation (18), sketch a Bessel function $J_{l+\frac{1}{2}}$ like we did in class; then the equation $J_{l+\frac{1}{2}}(\sqrt{\lambda}a) = 0$ says that $\sqrt{\lambda}a$ must equal one of the roots where the Bessel function crosses the axis. This gives a whole sequence of possible values for λ, and we call these values $\lambda_{l1}, \lambda_{l2}$, and so on.

Pages 261–262: Examples, and spherical harmonics

- Read Example 1. Then skim lightly down to "Spherical Harmonics".
- Read "Spherical Harmonics".

Math 442 — Day 38 Reading

Material: Sections 9.1, 9.2

Page 216: The characteristic cone

- Read page 216 and the first paragraph of page 217, which introduce you to the wave equation in 3D. The characteristic cone $|x - x_0| = c|t - t_0|$ is simply the extension to higher dimensions of the characteristic triangle in the xt-plane in Section 2.2.

Math 442 — Day 39 Reading

Material: Sections 9.2 and 9.3

Pages 226–227: Wave equation in 2D

- I will lecture on this Hadamard "method of descent".

Pages 233–234: Wave equation with a source, in 3D

Optional:

- The Duhamel formula (11) is formally just like in 1D.
- By substituting in our Kirchhoff solution formula, we get the solution (13) for the wave equation with a source in 3D.

Math 442 — Day 40 Reading

Material: Section 12.3

Page 325: Motivation

- Regard this interesting material as "formal" motivation (*i.e.* don't worry about justifying the steps).

Page 326: Basic formulas

- Regard (4) as the definition of the Fourier transform, so that (3) is the "Fourier inversion formula", giving you back f in terms of its transform F.
- The table on page 326 gives a number of useful transforms. Try using the definition (4) of the Fourier transform to verify the stated transform of the delta function, and of the square pulse function

$$H(a-|x|) = \begin{cases} 1 & \text{if } -a < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

(Here H is the Heaviside step function: $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \leq 0$.)

We will handle some of the other examples in class.

Page 327: Properties of the Fourier transform

- We will verify properties (i)–(vi) in class.
- Note that the Parseval identity (14) is analogous to the Parseval identity for Fourier series, formula (19) on page 128. In both cases, the norm squared on the space side equals the norm squared on the frequency (transform) side.

MATH 442 — DAY 41 READING

Material: Section 12.3 continued

Page 327: Heisenberg Uncertainty Principle

- The Uncertainty Principle is one of the key insights of 20th century physics. We will cover its proof, in class.

Page 328: Convolution

- Convolutions are important; for example, we solved the diffusion equation in Section 2.4 using convolution with a source function (formula (6) on page 47).
- In class we will cover the proof that the transform of $f * g$ is FG.

Math 442 — Day 42 Reading

Material: Section 12.4

Pages 330–331: Source function for the diffusion equation

- This material shows how to use the Fourier transform to solve PDEs: after Fourier transforming the x-variable, $\frac{\partial^2}{\partial x^2}$ turns into multiplication by $(ik)^2 = -k^2$. Thus the PDE gets transformed into just an ODE in the t-variable (for each fixed k).

Pages 331–332: Source function for the wave equation

- Omit this material.

Pages 332–333: Laplace's equation in a half-plane

- Why does Strauss say "We cannot transform the y variable"?
- Note that the calculation from (16) to (17) is essentially the same as result (7) on page 326, with some relabeling of the variables.
- *Exercise.* Express the solution (17) in polar coordinates.

Math 442 — Day 43 Reading

No reading. We will cover further material on transform methods.