

Math 442 — Day 22 Worksheet

Material: Section 5.1 – Solving diffusion and wave equations by Fourier series

Exercise 1. Diffusion Equation

(i) [**Dirichlet BC**] Solve the diffusion equation on $0 < x < \ell$ with Dirichlet BC and with IC $u(x, 0) = \phi(x)$. Assume ϕ can be expanded in a sine series.

Solution. By Separation of Variables (Section 4.1),

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \exp[-(n\pi/\ell)^2 kt] A_n \sin(n\pi x/\ell).$$

The initial condition is

$$\phi(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/\ell).$$

By Section 5.1 (coefficients of Fourier series), the coefficients are

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \sin(n\pi x/\ell) dx, \quad n \geq 1.$$

(ii) [**Neumann BC**] Solve the diffusion equation on $0 < x < \ell$ with Neumann BC and with IC $u(x, 0) = \phi(x)$. Assume ϕ can be expanded in a cosine series.

Solution.

(iii) [**Periodic BC**] Solve the diffusion equation on $-\ell < x < \ell$ with Periodic BC and with IC $u(x, 0) = \phi(x)$. Assume ϕ can be expanded in a full Fourier series.

Solution.

Exercise 2. Wave Equation

(i) [**Dirichlet BC**] Solve the wave equation on $0 < x < \ell$ with Dirichlet BC and with IC $u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$. Assume ϕ and ψ can be expanded in sine series.

Solution.

(ii) [**Neumann BC**] Solve the wave equation on $0 < x < \ell$ with Neumann BC and with IC $u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$. Assume ϕ and ψ can be expanded in cosine series.

Solution.

(iii) [**Periodic BC**] Solve the wave equation on $-\ell < x < \ell$ with Periodic BC and with IC $u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$. Assume ϕ and ψ can be expanded in full Fourier series.

Solution.