

Math 461 Fall 2006 — Test 1 Solutions

Total points: **100**. Do all questions. **Explain** all answers. No notes, books, or electronic devices.

1. [20 points] **Do part (a) or (b) but not both.**

(a) In how many ways can n identical balls be distributed into r urns so that: urn #1 contains at least m_1 balls, \dots , urn # r contains at least m_r balls. Assume that $n \geq \sum_{i=1}^r m_i$.

(b) A fair die is rolled 9 times. If at least one roll is known to be a “6”, then what is the probability that exactly two of the rolls are “6”?

Solution for (a). This is Chapter 2 Theoretical Exercise 20, one of the homework problems.

Solution for (b).

$$\begin{aligned} & P(\text{two “6”s} \mid \text{at least one “6”}) \\ &= \frac{P(\text{two “6”s and at least one “6”})}{P(\text{at least one “6”})} \\ &= \frac{P(\text{two “6”s})}{1 - P(\text{no “6”s})} \\ &= \frac{\binom{9}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^7}{1 - \left(\frac{5}{6}\right)^9}, \end{aligned}$$

where the binomial coefficient $\binom{9}{2}$ counts the ways to choose *which* two die rolls are to be “6” (Bernoulli trials!).

2. [20 points] A cruise ship has 10 Americans and 12 Brazilians as passengers. For dinner, a group of 8 passengers is selected at random to dine with the captain. Find the probability that at least one American and at least one Brazilian is selected.

Solution 1. The most straightforward method is to count all the possible groups, from 1 American and 7 Brazilians, ..., to 7 Americans and 1 Brazilian. In other words the probability is

$$\frac{\sum_{k=1}^7 \binom{10}{k} \binom{12}{8-k}}{\binom{22}{8}}.$$

Solution 2. Another method is to compute the complementary probability. If there is *not* “at least one American and at least one Brazilian selected” then the group must consist either of all Brazilians or all Americans. So

$$\begin{aligned} &P(\text{at least one American and at least one Brazilian}) \\ &= 1 - P(\text{all Brazilians or all Americans}) \\ &= 1 - P(\text{all Brazilians}) - P(\text{all Americans}) + P(\text{all Brazilians and all Americans}) \\ &= 1 - \frac{\binom{12}{8}}{\binom{22}{8}} - \frac{\binom{10}{8}}{\binom{22}{8}} + 0. \end{aligned}$$

You can check this equals the answer found in Solution 1.

An incorrect method. The formula

$$\frac{\binom{10}{1} \binom{12}{1} \binom{20}{6}}{\binom{22}{8}}$$

represents “choosing one American, choosing one Brazilian, and then choosing six more passengers from the remaining 20”. But this will **overcount** the possible outcomes. Suppose you choose an American A and a Brazilian B, then choose six more passengers, the Americans C,D,E,F,G,H. But you will count this situation again (many times), for example when you choose the American C and Brazilian B, then choose six more passengers, the Americans A,D,E,F,G,H.

Incidentally, the formula above evaluates to $(10 \cdot 12 \cdot 8 \cdot 7) / (22 \cdot 21)$, which is clearly greater than 1 and hence cannot represent a probability.

3. [20 points] **Do part (a) or (b) but not both.**

(a) An 8-card hand is drawn (without replacement) from a standard deck of 52 cards. Find the probability that the hand contains the ace and king in at least one suit (*e.g.* AK-spades).

(b) Compute the probability that a bridge hand is void in at least one suit.

Extra credit: Explain what is wrong with the reasoning that leads to the incorrect answer $\binom{4}{1} \binom{39}{13} / \binom{52}{13}$.

Solution for (a). Write S, C, D, H for the events that the hand contains the ace and king of Spades, Clubs, Diamonds, Hearts (respectively). Then by the Inclusion–Exclusion Principle,

$$\begin{aligned} & P(S \cup C \cup D \cup H) \\ &= P(S) + P(C) + P(D) + P(H) - P(SC) - P(SD) - \dots \\ &\quad + P(SCD) + P(SCH) + \dots - P(SCDH) \\ &= \binom{4}{1} \frac{\binom{50}{6}}{\binom{52}{8}} - \binom{4}{2} \frac{\binom{48}{4}}{\binom{52}{8}} + \binom{4}{3} \frac{\binom{46}{2}}{\binom{52}{8}} - \binom{4}{4} \frac{\binom{44}{0}}{\binom{52}{8}}. \end{aligned}$$

By the way, the *complementary* probability doesn't seem to help on this problem, because it is not clear how to evaluate the probability of a hand not containing the ace and king in any suit. (Such hands could contain aces and kings, of course, just not in the same suit.)

An incorrect method. The formula

$$\frac{\binom{4}{1} \binom{50}{6}}{\binom{52}{8}}$$

represents “choosing one suit, taking the ace and king in that suit, and then choosing six more cards from the remaining 50”. But this will **overcount** the possible outcomes. Suppose you choose the suit Hearts, take the ace and king of Hearts, and then choose six more cards, including the ace and king of Clubs. You will count this situation again, for example, when you choose the suit Clubs, take the ace and king of Clubs, and then choose choose six more cards including the ace and king of Hearts.

Solution for (b). This is Chapter 2 Exercise 54, one of the homework problems.

Write S, C, D, H for the events that the hand is void in Spades, Clubs, Diamonds, Hearts (respectively). Then by the Inclusion–Exclusion Principle,

$$\begin{aligned} & P(S \cup C \cup D \cup H) \\ &= P(S) + P(C) + P(D) + P(H) - P(SC) - P(SD) - \dots \\ &\quad + P(SCD) + P(SCH) + \dots - P(SCDH) \\ &= \binom{4}{1} \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3} \frac{\binom{13}{13}}{\binom{52}{13}} - 0, \end{aligned}$$

where the “0” is because the hand can’t be void in all four suits.

By the way, the *complementary* probability can be used on this problem (how?!), but it gives a much more complicated formula for the answer.

Extra credit: The formula $\binom{4}{1} \frac{\binom{39}{13}}{\binom{52}{13}}$ suggests that we should choose one of the four suits, and arrange to be void in that suit by choosing 13 of the remaining 39 cards. But when we choose those 13 cards, we might end up void in another suit as well. So for example, we might choose to be void in Spades, but end up void in Clubs as well. This outcome will be *overcounted* by our formula, because when we choose to be void in Clubs, we might end up void in Spades as well. The overcounting makes the formula an incorrect answer to the problem. We remove the overcounting by using the Inclusion–Exclusion formula.

4. [20 points] In a certain town, 6% of people are sick with the flu, another 14% are infected (but not yet sick), and the other 80% are healthy. None of the sick people feel cheerful, half of the infected people feel cheerful, and three quarters of the healthy people feel cheerful.

Find the probability that a cheerful person is healthy.

Solution 1. Write S, I, H for the events of being sick, infected or healthy, and C for the event of being cheerful. The events S, I, H are mutually exclusive, and their union is the whole sample space (since $6\%+14\%+80%=100\%$). Hence *Bayes' Formula* applies, and says the probability that a person is healthy, given they are cheerful, is

$$\begin{aligned} P(H|C) &= \frac{P(HC)}{P(C)} \\ &= \frac{P(C|H)P(H)}{P(C|S)P(S) + P(C|I)P(I) + P(C|H)P(H)} \\ &= \frac{(0.75)(0.80)}{(0)(0.06) + (0.5)(0.14) + (0.75)(0.80)} \\ &= \frac{60}{67}. \end{aligned}$$

Solution 2. Alternatively, one can organize the calculation in a table, or in words as follows. Out of every 100 people, the number of cheerful people is $\frac{1}{2} \cdot 14 + \frac{3}{4} \cdot 80 = 67$. Of these 67, we know that $\frac{3}{4} \cdot 80 = 60$ are healthy. Thus the probability that a person is healthy, given they are cheerful, is $60/67$.

5. [20 points] Let $0 < p < 1$. Two baseball teams called the Cubs and the Sox play in the World Series. In each game, the Cubs win with probability p , and the Sox win with probability $1 - p$, and the games are independent of one another. The first team to win 4 games wins the series.

Find the probability that the Cubs win the series after 5 games.

Solution. This is a Bernoulli trial situation: we imagine that the teams play 7 games (even though in practice they stop after one team wins 4 games). Then

$$\begin{aligned} &P(\text{Cubs win the series after 5 games}) \\ &= P(\text{Cubs win 3 of the first 4 games, and then win the 5th game}) \\ &= \binom{4}{3} p^3 (1 - p) \cdot p, \end{aligned}$$

where the binomial coefficient $\binom{4}{3}$ represents the number of ways to choose *which* 3 of the first 4 games are won by the Cubs. (It would be incorrect to use $\binom{5}{4}$, because that would count the situation where the Cubs win the first 4 games and lose the fifth game, in which case the Cubs would win the series after 4 games rather than 5.)

6. [*Extra credit*, 15 points] Consider $k+1$ coins with the property that coin # i has probability i/k of coming up heads (for the $k+1$ values of $i = 0, 1, \dots, k$). A coin is randomly selected and then repeatedly tossed.

If the first n tosses are all heads, then what is the probability the next toss will be heads? Then show that when k is large, your answer equals approximately $(n+1)/(n+2)$.

Solution. This is “Laplace’s Problem of Succession”, which you can read about on page 108 in the textbook.