Math 545 Homework 2

Due: Monday 10 November by 5pm, to my office (376 Altgeld Hall)

Homework collaboration sessions:

Tuesday 28 October, 5-6pm, in 347 AH.

Thursday 6 November, 3-4pm, in 241 AH.

Come and work together.

Problem 1 (Hilbert transform of indicator function). Put $f = \mathbb{1}_{[a,b]}$ where $[a, b] \subset (-\pi, \pi)$ is a closed interval.

(a) Show the Hilbert transform $(Hf)(t)$ blows up logarithmically at the endpoints a and b. (*Hint.* $\cot(\frac{1}{2}t) = \frac{2}{t} +$ (bounded function), for $t \in [-\pi, \pi]$.)

(b) Conclude that the Hilbert transform on $\mathbb{T}$ is not strong (∞, ∞).

Problem 2 (Fourier synthesis on ℓ^p). Let $1 \leq p \leq 2$.

Prove that the Fourier synthesis operator T, defined by

$$(T\{c_n\})(t) = \sum_{n\in\mathbb{Z}} c_n e^{int},$$

is bounded from $\ell^p(\mathbb{Z})$ to $L^{p'}(\mathbb{T})$. Estimate the norm of T.

Extra credit. Show the series converges unconditionally, in $L^{p'}(\mathbb{T})$.

Problem 3 (Parseval on L^p). Do part (a) or part (b). You may do both parts if you wish.

(a) Let $1 \leq p \leq 2$. Take $f \in L^p(\mathbb{T})$ and $g \in L^1(\mathbb{T})$ with $\{\widehat{g}(n)\} \in \ell^p(\mathbb{Z})$. Prove that $g \in L^{p'}(\mathbb{T})$, and establish the Parseval identity

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{g(t)}\, dt = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

(In your solution, explain why the integral and sum are absolutely convergent.)

(b) Let $1 < p < \infty$. Take $f \in L^p(\mathbb{T})$ and $g \in L^{p'}(\mathbb{T})$. Prove the Parseval identity

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{g(t)}\, dt = \lim_{N\to\infty} \sum_{|n|\leq N} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

Problem 4 (Fourier analysis into a weighted space). Let $1 < p \le 2$.

(a) Show

$$\left(\sum_{n\neq 0} |\widehat{f}(n)|^p |n|^{p-2}\right)^{1/p} \le C_p \|f\|_{L^p(\mathbb{T})} \qquad \text{for all } f \in L^p(\mathbb{T}).$$

Hint. $Y = \mathbb{Z} \setminus \{0\}$ with $\nu =$ counting measure weighted by n^{-2}.

(b) Show that combining the Hölder and Hausdorff–Young inequalities in the obvious way does *not* prove part (a).

Problem 5 (Poisson extension). Recall P_r denotes the Poisson kernel on $\mathbb{T}$, and write $\mathbb{D}$ for the open unit disk in the complex plane. Suppose $f \in C(\mathbb{T})$ and define

$$v(re^{it}) = \begin{cases} (P_r * f)(t) & \text{for } 0 \le r < 1,\ t \in \mathbb{T}, \\ f(t) & \text{for } r = 1,\ t \in \mathbb{T}, \end{cases}$$

so that v is defined on the closed disk $\overline{\mathbb{D}}$.

(a) Show v is C^∞ smooth and harmonic ($\Delta v = 0$) in $\mathbb{D}$.

(b) Show v is continuous on $\overline{\mathbb{D}}$.

(c) [Optional; no credit] Assume $f \in C^\infty(\mathbb{T})$ and show $v \in C^\infty(\overline{\mathbb{D}})$. (Parts (a) and (b) show v is smooth on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Thus the task is to prove each partial derivative of v on $\mathbb{D}$ extends continuously to $\overline{\mathbb{D}}$.).

Aside. $(P_r * f)(t)$ is called the *harmonic extension* to the disk of the boundary function f.

Problem 6 (Boundary values lose half a derivative). Assume u is a smooth, real-valued function on a neighborhood of $\overline{\mathbb{D}}$, and define

$$f(t) = u(e^{it})$$

for the boundary value function of u. Hence $f \in C^\infty(\mathbb{T})$, and so the Poisson extension v belongs to $C^\infty(\overline{\mathbb{D}})$ by Problem 5(c).

(a) Prove

$$\frac{1}{2\pi} \int_{\mathbb{D}} |\nabla v|^2 \, dA = \sum_{n\in\mathbb{Z}} |n| |\widehat{f}(n)|^2.$$

Hint. Use one of Green's formulas, and remember $v = \overline{v}$ since f and v are real-valued.

(b) Prove

$$\int_{\mathbb{D}} |\nabla v|^2 \, dA \le \int_{\mathbb{D}} |\nabla u|^2 \, dA.$$

Hint. Write $u = v + (u - v)$ and use one of Green's formulas.

Aside. This result is known as "Dirichlet's principle". It asserts that among all functions having the same boundary values, the harmonic function has smallest Dirichlet integral. As your proof reveals, this result holds on arbitrary domains.

(c) Conclude

$$\sum_{n \in \mathbb{Z}} |n||\widehat{f}(n)|^2 \le \frac{1}{2\pi} \int_{\mathbb{D}} |\nabla u|^2 \, dA.$$

Discussion. We say f has "half a derivative" in L^2, since $\{|n|^{1/2}\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$. Justification: if f has zero derivatives ($f \in L^2(\mathbb{T})$) then $\{\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$, and if f has one derivative ($f' \in L^2(\mathbb{T})$) then $\{n\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$. Halfway inbetween lies the condition $\{|n|^{1/2}\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$.

Boundary trace inequalities like in part (c) are important for partial differential equations and Sobolev space theory. The inequality says, basically, that if a function u has one derivative ∇u belonging to L^2 on a domain, then u has half a derivative in L^2 on the boundary. Thus the boundary value loses half a derivative, compared to the original function.

Note that in this problem, $f \in C^\infty(\mathbb{T})$ and so certainly $f' \in L^2(\mathbb{T})$, which implies $\{n\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$. You might wonder, then, why you should bother proving the weaker result $\{|n|^{1/2}\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$ in part (c). But actually you prove more in part (c): you obtain a *norm estimate* on $\{|n|^{1/2}\widehat{f}(n)\} \in \ell^2(\mathbb{Z})$ in terms of the L^2 norm of ∇u. (We do not have such a norm estimate on $\{n\widehat{f}(n)\}$.) This norm estimate means that the restriction map

$$\begin{aligned} H^1(\mathbb{D}) &\to H^{1/2}(\partial\mathbb{D}) \\ u &\mapsto f \end{aligned}$$

is bounded from the Sobolev space $H^1(\mathbb{D})$ on the disk with one derivative in L^2 to the Sobolev space $H^{1/2}(\partial\mathbb{D})$ on the boundary circle with half a derivative in L^2.

Aside. The notion of fractional derivatives defined via Fourier coefficients can be extended to fractional derivatives in $\mathbb{R}^d$, by using Fourier transforms.

Problem 7 (Measuring diameters of stars).

Enjoyable reading; nothing to hand in.

Read the attached Chapter 95 from T. W. Körner "Fourier Analysis", which shows how the diameters of stars can be estimated using Fourier transforms of radial functions, and convolutions.

T. W. Körner "Fourier Analysis"

95

THE DIAMETER OF STARS

We are used to thinking of stars as being so far away as to act as point sources of light. The fact that they appear to us as 'twinkling' patches of light is due to atmospheric effects. But, surprising as it may seem to a layman, the nearest stars are sufficiently close that, if it were not for the effects of the atmosphere, a good photograph using a good telescope would show them as tiny discs. Since observations of the nearest stars at six-monthly intervals (i.e. using a diameter of the earth's orbit as a surveyor's base line) enable astronomers to measure the distance of these stars, knowledge of the apparent diameter (i.e. the diameters of the discs on the photographic plate) would then enable us to calculate the true diameters of the nearest stars.

However, the blurring due to atmospheric effects is much greater than the apparent diameter we wish to observe. How can we get round this problem? Soon we will be able to use the 'big science' method and spend our way out of trouble by putting our telescope in orbit above the atmosphere. A more elegant (and considerably cheaper) solution has been found by Labeyrie.

Suppose we photograph a point source at time t and suppose that, without atmospheric effects, it would appear at $\mathbf{0}$ on our photographic plate. Owing to atmospheric effects we obtain a picture whose 'brightness' at a point $\mathbf{x}$ on the plate is $\lambda K_t(\mathbf{x})$ (where λ is the 'brightness' of our original point source). Suppose now that we have a second point source which has 'brightness' λ' and would (in the absence of atmospheric effects) appear as a point at $\mathbf{y}'$ on the plate. If $\|\mathbf{y}'\|$ is very small (i.e. the second point source is very close to the first in the sky) we would expect light coming from it to be affected in the same way as for the first and so we expect a picture with 'brightness' $\lambda' K_t(\mathbf{x}-\mathbf{y}')$.

More generally, if we had point sources of 'brightness' $\lambda_1, \lambda_2, \ldots, \lambda_n$ whose images would (in the absence of atmospheric effects) lie at $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ with $\|\mathbf{y}_1\|$, $\|\mathbf{y}_2\|, \ldots, \|\mathbf{y}_n\|$ very small we would expect an image of the form

$$\sum_{j=1}^{n} \lambda_j K_t(\mathbf{x}-\mathbf{y}_j).$$

Replacing the sum by an integral in the usual way we would expect that a 'true image' $f(\mathbf{x})$ would be replaced by an 'actual image'

$$\iint f(\mathbf{y}) K_t(\mathbf{x}-\mathbf{y})\, dA(\mathbf{y})$$

(provided $f(\mathbf{y}) = 0$ except when $\|\mathbf{y}\|$ is very small). In other words, if in the absence of atmospheric effects, we would get an image f, then in the presence of atmospheric effects, we will get an image $f * K_t$ where K_t represents the blurring effect of the atmosphere.

The reader should note that K_t is not fixed but (since it is due to random atmospheric effects) itself varies randomly in time. Thus our problem appears to be: 'Extract f from $f * K_t$ where K_t is an unknown random function' and so appears to be insoluble. The first idea that comes to mind is probably that of 'averaging' the image received at various times $t(1), t(2), \ldots, t(n)$, say, to obtain

$$n^{-1} \sum_{j=1}^{n} f * K_{t(j)} = f * \sum_{j=1}^{n} n^{-1} K_{t(j)} = f * \tilde{K},$$

where $\tilde{K} = n^{-1} \sum_{j=1}^{n} K_{t(j)}$ is the 'average blur'. But we know little more about the 'average blur' $\tilde{K}$ than we know about its random constituents.

Labeyrie's idea is to take the Fourier transform of our image $\phi_t = f * K_t$. Since convolution becomes multiplication under Fourier transforms, $\hat{\phi}_t = \hat{f} \hat{K}_t$ and so the zeros of $\hat{\phi}_t$ are the zeros of $\hat{f}$ and of $\hat{K}_t$. Of course for one photograph we cannot distinguish the zeros of $\hat{f}$ and those of $\hat{K}_t$. But since K_t is random, so are the zeros of $\hat{K}_t$! Thus if we take a sequence $\hat{\phi}_{t(1)}, \hat{\phi}_{t(2)}, \ldots, \hat{\phi}_{t(n)}$ the only zeros of the $\hat{\phi}_{t(j)}$ common to all the j will be those of $\hat{f}$, the other zeros being the random ones of each random $\hat{K}_{t(j)}$. In particular, if we superimpose pictures of the $|\hat{\phi}_{t(j)}|$ to form

$$\psi(\zeta) = \sum_{j=1}^{n} |\hat{\phi}_{t(j)}(\zeta)| = \sum_{j=1}^{n} |\hat{f}(\zeta)|\,|\hat{K}_{t(j)}(\zeta)|$$

the zeros of $\hat{f}(\zeta)$ will stand out clearly as the zeros of $\psi(\zeta)$.

Of course the zeros of $\hat{f}$ do not suffice to determine f in general any more than the zeros of a function g determine g. (Observe, for example, that $\hat{f}$ and $(f * f)^\wedge$ have the same zeros.) However, we are not dealing with an arbitrary function f but with the 'true' image of a star which we expect to be a uniform disc. In other words, we expect

$$f(\mathbf{x}) = \lambda D(\varepsilon^{-1}(\mathbf{x}-\mathbf{x}_0)),$$

where

$$D(\mathbf{x}) = 1 \quad \text{for } |\mathbf{x}| \leqslant 1,$$
$$D(\mathbf{x}) = 0 \quad \text{for } |\mathbf{x}| \geqslant 1,$$

ε is the radius of the image, $\mathbf{x}_0$ its centre and λ some positive real number. We know neither $\mathbf{x}_0$ nor ε and we wish to find ε.

The reader already knows the formulae which enable us to obtain $\hat{f}$ in terms of $\hat{D}$ but it can do no harm to rederive them. In effect, making the substitutions $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{w} = \varepsilon^{-1}\mathbf{y}$ we obtain

$$\begin{aligned}\hat{f}(\zeta) &= \iint_{\mathbb{R}^2} \lambda D(\varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0)) \exp(-i\zeta\cdot\mathbf{x})\, dA(\mathbf{x}) \\ &= \iint_{\mathbb{R}^2} \lambda D(\varepsilon^{-1}\mathbf{y}) \exp(-i\zeta\cdot(\mathbf{y} + \mathbf{x}_0))\, dA(\mathbf{y}) \\ &= \lambda \exp(-i\zeta\cdot\mathbf{x}_0) \iint_{\mathbb{R}^2} D(\varepsilon^{-1}\mathbf{y}) \exp(-i\zeta\cdot\mathbf{y})\, dA(\mathbf{y}) \\ &= \lambda\varepsilon^2 \exp(-i\zeta\cdot\mathbf{x}_0) \iint_{\mathbb{R}^2} D(\mathbf{w}) \exp(-i\varepsilon\zeta\cdot\mathbf{w})\, dA(\mathbf{w}) \\ &= \lambda\varepsilon^2 \exp(-i\zeta\cdot\mathbf{x}_0)\hat{D}(\varepsilon\zeta).\end{aligned}$$

Thus the zeros of $\hat{f}$ are given by $\hat{D}(\varepsilon\zeta) = 0$.

Since D is radially symmetric, so is $\hat{D}$. Thus the zeros of $\hat{D}$ will appear in rings and so the zeros of $\hat{f}$ will appear in rings whose spacing is inversely proportional to the required radius ε. (Thus the further apart the rings, the smaller the radius of the 'true' image.)

In order to compute ε we need to know the location of the rings of zeros of $\hat{D}$. Since we could compute $\hat{D}$ numerically in the same way as we will have to compute $\hat{\phi}_{t(j)}$ this presents no problems. The reader may, however, be interested to see an explicit formula for $\hat{D}$ in terms of Bessel functions. (This expression may also quieten the fears of any reader who has observed that, whilst the argument above shows that the zeros, if any, of $\hat{D}$ form circular rings, it does not show that $\hat{D}$ actually has any zeros.)

Lemma 95.1. *Suppose $D:\mathbb{R}^2 \to \mathbb{R}$ is given by*

$$\begin{aligned} D(\mathbf{x}) &= 1 \quad \textit{for } \|\mathbf{x}\| \leqslant 1, \\ D(\mathbf{x}) &= 0 \quad \textit{otherwise.} \end{aligned}$$

Then $\hat{D}(\zeta) = 2\pi J_1(\|\zeta\|)/\|\zeta\|$ where J_1 is the first Bessel function.

Remark. Our proof makes use of the relations

$$\int_0^{2\pi} \exp(-ia\cos\theta)\, d\theta = 2\pi J_0(a), \tag{1}$$

$$\int_0^x y^n J_{n-1}(y)\, dy = x^n J_n(x), \tag{2}$$

which may be found proved in most standard mathematical methods texts.

Proof. By definition

$$\hat{D}(\zeta) = \iint D(\mathbf{x}) \exp -i\zeta\cdot\mathbf{x}\, dA(\mathbf{x}).$$

Choosing polar coordinates (r, θ) in such a way that ζ has coordinates $(\zeta, 0)$ we see that

$$\hat{D}(\zeta) = \int_0^1 \int_0^{2\pi} r \exp -(i\zeta r\cos\theta)\, d\theta\, dr.$$

Making use of relation (1) above and making the substitution $s = \zeta r$ we obtain

$$\hat{D}(\zeta) = 2\pi \int_0^1 rJ_0(\zeta r)\, dr = \frac{2\pi}{\zeta^2} \int_0^\zeta sJ_0(s)\, ds.$$

Thus by relation (2)

$$\hat{D}(\zeta) = 2\pi J_1(\zeta)/\zeta,$$

which is the required result. (The case $\zeta = 0$ must be considered separately but here, trivially, $\hat{D}(\mathbf{0}) = 2\pi$.) ■

Just as in the previous chapter we have sketched a method which requires the numerical computation of several Fourier transforms and the reconstitution of a function f from a limited knowledge of $\hat{f}$ (in this case the location of its zeros) and other information about the form of f (in this case that f is a (scalar multiple of) the characteristic (indicator) function of a disc). The method of Labeyrie's gives results consistent with Michelson's interferometer results but because of its simplicity has been applied to many more stars. (Over 30 have now had their diameters measured in this way.)