Math 545 Homework 2

Due: Monday 10 November by 5pm, to my office (376 Altgeld Hall)

Homework collaboration sessions:
   Tuesday 28 October, 5-6pm, in 347 AH.
   Thursday 6 November, 3-4pm, in 241 AH.
Come and work together.

Problem 1 (Hilbert transform of indicator function). Put \( f = \mathbb{1}_{[a,b]} \) where \([a, b] \subset (-\pi, \pi)\) is a closed interval.
   (a) Show the Hilbert transform \( (Hf)(t) \) blows up logarithmically at the endpoints \( a \) and \( b \). (Hint. \( \cot(\frac{1}{2}t) = \frac{1}{t} + \) (bounded function), for \( t \in [-\pi, \pi] \).
   (b) Conclude that the Hilbert transform on \( \mathbb{T} \) is not strong \((\infty, \infty)\).

Problem 2 (Fourier synthesis on \( \ell^p \)). Let \( 1 \leq p \leq 2 \).
Prove that the Fourier synthesis operator \( T \), defined by
\[
(T\{c_n\})(t) = \sum_{n \in \mathbb{Z}} c_n e^{int},
\]
is bounded from \( \ell^p(\mathbb{Z}) \) to \( L^p(\mathbb{T}) \). Estimate the norm of \( T \).

Extra credit. Show the series converges unconditionally, in \( L^p(\mathbb{T}) \).

Problem 3 (Parseval on \( L^p \)). Do part (a) or part (b). You may do both parts if you wish.
   (a) Let \( 1 \leq p \leq 2 \). Take \( f \in L^p(\mathbb{T}) \) and \( g \in L^1(\mathbb{T}) \) with \( \{\hat{g}(n)\} \in \ell^p(\mathbb{Z}) \).
   Prove that \( g \in L^p(\mathbb{T}) \), and establish the Parseval identity
\[
\frac{1}{2\pi} \int_{\mathbb{T}} f(t)\overline{g(t)} \, dt = \sum_{n \in \mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.
\]
(In your solution, explain why the integral and sum are absolutely convergent.)

   (b) Let \( 1 < p < \infty \). Take \( f \in L^p(\mathbb{T}) \) and \( g \in L^p(\mathbb{T}) \). Prove the Parseval identity
\[
\frac{1}{2\pi} \int_{\mathbb{T}} f(t)\overline{g(t)} \, dt = \lim_{N \to \infty} \sum_{|n| \leq N} \hat{f}(n)\overline{\hat{g}(n)}.
\]
Problem 4 (Fourier analysis into a weighted space). Let $1 < p \leq 2$.
(a) Show
\[
\left( \sum_{n \neq 0} |\hat{f}(n)|^p |n|^{p-2} \right)^{1/p} \leq C_p \|f\|_{L^p(\mathbb{T})} \quad \text{for all } f \in L^p(\mathbb{T}).
\]

*Hint.* $Y = \mathbb{Z} \setminus \{0\}$ with $\nu = $ counting measure weighted by $n^{-2}$.

(b) Show that combining the Hölder and Hausdorff–Young inequalities in the obvious way does *not* prove part (a).

Problem 5 (Poisson extension). Recall $P_r$ denotes the Poisson kernel on $\mathbb{T}$, and write $\mathbb{D}$ for the open unit disk in the complex plane. Suppose $f \in C(\mathbb{T})$ and define
\[
\nu(re^{it}) = \begin{cases} 
(P_r * f)(t) & \text{for } 0 \leq r < 1, \ t \in \mathbb{T}, \\
\hat{f}(t) & \text{for } r = 1, \ t \in \mathbb{T},
\end{cases}
\]
so that $\nu$ is defined on the closed disk $\overline{\mathbb{D}}$.

(a) Show $\nu$ is $C^\infty$ smooth and harmonic ($\Delta \nu = 0$) in $\mathbb{D}$.

(b) Show $\nu$ is continuous on $\overline{\mathbb{D}}$.

(c) [Optional; no credit] Assume $f \in C^\infty(\mathbb{T})$ and show $\nu \in C^\infty(\overline{\mathbb{D}})$. (Parts (a) and (b) show $\nu$ is smooth on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Thus the task is to prove each partial derivative of $\nu$ on $\mathbb{D}$ extends continuously to $\overline{\mathbb{D}}$.)

*Aside.* $(P_r * f)(t)$ is called the harmonic extension to the disk of the boundary function $f$.

Problem 6 (Boundary values lose half a derivative). Assume $u$ is a smooth, real-valued function on a neighborhood of $\overline{\mathbb{D}}$, and define
\[
f(t) = u(e^{it})
\]
for the boundary value function of $u$. Hence $f \in C^\infty(\mathbb{T})$, and so the Poisson extension $\nu$ belongs to $C^\infty(\overline{\mathbb{D}})$ by Problem 5(c).

(a) Prove
\[
\frac{1}{2\pi} \int_{\mathbb{D}} |\nabla \nu|^2 \, dA = \sum_{n \in \mathbb{Z}} |n| |\hat{f}(n)|^2.
\]

*Hint.* Use one of Green's formulas, and remember $\nu = \overline{v}$ since $f$ and $\nu$ are real-valued.
(b) Prove
\[ \int_D |\nabla v|^2 dA \leq \int_D |\nabla u|^2 dA. \]

*Hint.* Write \( u = v + (u - v) \) and use one of Green’s formulas.

*Aside.* This result is known as “Dirichlet’s principle”. It asserts that among all functions having the same boundary values, the harmonic function has smallest Dirichlet integral. As your proof reveals, this result holds on arbitrary domains.

(c) Conclude
\[ \sum_{n \in \mathbb{Z}} |n||\hat{f}(n)||^2 \leq \frac{1}{2\pi} \int_D |\nabla u|^2 dA. \]

*Discussion.* We say \( f \) has “half a derivative” in \( L^2 \), since \( \{|n|^{1/2} \hat{f}(n)| \in l^2(\mathbb{Z}) \). Justification: if \( f \) has zero derivatives \( f \in L^2(\mathbb{T}) \) then \( \{\hat{f}(n)\} \in l^2(\mathbb{Z}) \), and if \( f \) has one derivative \( f' \in L^2(\mathbb{T}) \) then \( \{n\hat{f}(n)\} \in l^2(\mathbb{Z}) \). Halfway in between lies the condition \( \{|n|^{1/2} \hat{f}(n)| \in l^2(\mathbb{Z}) \).

*Boundary trace* inequalities like in part (c) are important for partial differential equations and Sobolev space theory. The inequality says, basically, that if a function \( u \) has one derivative \( \nabla u \) belonging to \( L^2 \) on a domain, then \( u \) has half a derivative in \( L^2 \) on the boundary. Thus the boundary value loses half a derivative, compared to the original function.

Note that in this problem, \( f \in C^\infty(\mathbb{T}) \) and so certainly \( f' \in L^2(\mathbb{T}) \), which implies \( \{n\hat{f}(n)\} \in l^2(\mathbb{Z}) \). You might wonder, then, why you should bother proving the weaker result \( \{|n|^{1/2} \hat{f}(n)| \in l^2(\mathbb{Z}) \) in part (c). But actually you prove more in part (c): you obtain a norm estimate on \( \{|n|^{1/2} \hat{f}(n)| \in l^2(\mathbb{Z}) \) in terms of the \( L^2 \) norm of \( \nabla u \). (We do not have such a norm estimate on \( \{n\hat{f}(n)\}. \)) This norm estimate means that the restriction map

\[ H^1(\mathbb{D}) \to H^{1/2}(\partial \mathbb{D}) \]

\[ u \mapsto f \]

is bounded from the Sobolev space \( H^1(\mathbb{D}) \) on the disk with one derivative in \( L^2 \) to the Sobolev space \( H^{1/2}(\partial \mathbb{D}) \) on the boundary circle with half a derivative in \( L^2 \).

*Aside.* The notion of fractional derivatives defined via Fourier coefficients can be extended to fractional derivatives in \( \mathbb{R}^d \), by using Fourier transforms.
Problem 7 (Measuring diameters of stars).
Enjoyable reading; nothing to hand in.
Read the attached Chapter 95 from T. W. Körner “Fourier Analysis”,
which shows how the diameters of stars can be estimated using Fourier transforms of radial functions, and convolutions.
The diameter of stars

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Tom Keiser, F.W. Almgren's
which may be found proved in most standard mathematical methods texes.

\[ \int_{0}^{\pi} \int_{0}^{\theta} \sin^2 \varphi \cos^2 \varphi \, d\varphi \, d\theta = \frac{\theta^2}{2} \]

Remark: Our proof makes use of the relations.

Then \( g(\varphi) = \frac{\pi}{2} (\varphi^2 + \cos^2 \varphi) \).

Lemman 954. Suppose \( g(x) \) is given by

\[
\int_{0}^{\pi} g(x) \sin \varphi \, d\varphi = \int_{0}^{\pi} \cos \varphi \, d\varphi.
\]

The reader should prove this result on their own.

\[
(\varphi) = \int_{0}^{\pi} g(x) \sin \varphi \, d\varphi
\]

This is typically the case when \( g(x) \) must be considered separately.

The case \( \varphi = 0 \) must be considered separately.

Therefore, the required result is obtained.

Thus by relation (2)

\[
\int_{0}^{\pi} g(x) \sin \varphi \, d\varphi = \int_{0}^{\pi} \cos \varphi \, d\varphi.
\]

To obtain a useful form of the result, we expand the substitution \( \varphi = \int_{0}^{\pi} g(x) \sin \varphi \, d\varphi \).

Choosing polar coordinates \((\rho, \theta, \varphi)\) in such a way that has coordinates \((\theta, \varphi)\) we get

\[
(\varphi) = \int_{0}^{\pi} g(x) \sin \varphi \, d\varphi.
\]

Proof. By definition.

The diameter of area

\[
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\]