Math 553 — Fall 2004 — Final Exam

Total points: 175. Show ALL your working and make your explanations as full as possible. Electronic devices are not allowed on this exam; neither are books or notes.

Green Formulas:

$$\begin{aligned}\int_\Omega [v\Delta u + \nabla v \cdot \nabla u]\, dx &= \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\, dS \\ \int_\Omega [v\Delta u - u\Delta v]\, dx &= \int_{\partial\Omega} \left[v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right] dS\end{aligned}$$

Darboux Formula:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_u(x,r) = M_{\Delta u}(x,r)$$

1: (35 points) *First order equations*

Let $G(z) = \frac{1}{3}(1-z)^3$.

(a) Solve the conservation law $G(u)_x + u_y = 0$ for $x \in \mathbb{R}, 0 < y < 3$, given initial data

$$u(x,0) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(A well–labelled sketch of the characteristics is a good way to present your solution.)

(b) Then find an equation for the shock slope $x'(y)$ when $y > 3$.

2: (30 points) *Heat equation*

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$, with smooth boundary. Assume $u(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$ solves the following initial value problem for the heat equation:

$$\begin{aligned} u_t &= \Delta u, & x \in \Omega, \quad t > 0, \\ u(x,0) &= g(x), & x \in \Omega, \\ u(x,t) &= 0, & x \in \partial\Omega, \quad t > 0, \end{aligned}$$

where $g \in C_0^\infty(\Omega)$ is given, $g \not\equiv 0$. Assume $g(x) \leq 0$ for all $x \in \Omega$.

(a) Show that $u(x,t) \leq 0$ for all $x \in \Omega, 0 < t < T$.

(b) Actually $u(x,t) < 0$ for all $x \in \Omega, 0 < t < T$. Assuming this, explain why it means the heat equation allows “infinite propagation speed” of disturbances.

3: (15 points) *Nonhomogeneous wave equation in one dimension*

Use Duhamel's principle to find an explicit solution of

$$\begin{aligned} u_{tt} &= u_{xx} + e^x, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= 0. \end{aligned}$$

4: (30 points) Consider the following two functions on $\mathbb{R}^2$:

$$u(x,y) = \begin{cases} 2x^2 + 5 & \text{for } \; x > 0 \\ 0 & \text{for } \; x < 0 \end{cases}$$

$$v(x,y) = \begin{cases} x^2 + y^2 + 5 & \text{for } \; x > 0 \\ 0 & \text{for } \; x < 0 \end{cases}$$

Show one of these functions is a weak solution of the equation $w_{xy} = 0$, and the other function is not.

Hint. Start by writing out the definition of a weak solution.

5: (35 points) Let f be a smooth function with compact support in $\mathbb{R}^3$.

(a) Write down the fundamental solution of the Laplacian (solving $\Delta K = \delta$ in $\mathbb{R}^3$ where δ is the Dirac delta function).

(b) Write down a formula for $u(x)$ solving Poisson's equation $\Delta u = f$ in $\mathbb{R}^3$ with $u(x) \to 0$ as $|x| \to \infty$.

(c) Show formally that $\Delta u = f$ weakly.

(d) See next page.

[5. continued] (d) Assume f is radially symmetric and is supported in the unit ball. (Radially symmetric means that $f(x) = F(r)$ for some function F, where $|x| = r$.) Suppose also that $\int_{\mathbb{R}^3} f(x)\, dx = 1$.

Show $u(x) = K(x)$ for all $|x| > 1$.

Hint. Spherical coordinates in the formula for u.

Aside. Part (d) means physically that the gravitational potential of a radially symmetric planet is the same (outside the planet) as if all the mass were concentrated at the center.

6: (30 points) Consider Poisson's equation $\Delta u = f$ in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, with boundary condition $u = g$ on $\partial\Omega$.

State and prove a result about "continuous dependence of u on f and g".

Hint. Maximum principle. And if you cannot solve the problem as stated, then for half-credit on the problem you can assume $f \equiv 0$.

SOLUTIONS

1. The PDE is a first order conservation law, and so u is constant along the characteristics. The projected characteristics have the form $x = G'(u)y + x_0$. Since $G'(z) = -(1-z)^2$ and $G'(0) = -1$, we find

$$(G')^{-1}(w) = 1 - \sqrt{-w}, \qquad w \leq 0.$$

Since

$$G'(0) = -1 \quad \text{and} \quad G'(1) = 0,$$

we can immediately sketch most of the characteristics:

A fan starts at $(0,0)$ and a shock starts at $(1,0)$. By the solution to Riemann's problem, in the fan we have

$$u = (G')^{-1}(x/y) = 1 - \sqrt{-x/y}.$$

By the jump condition, the shock has slope

$$x'(y) = \frac{G(u_\ell) - G(u_r)}{u_\ell - u_r} = \frac{G(1) - G(0)}{1 - 0} = -\frac{1}{3}$$

and so the shock path is $x = 1 - y/3$, at least up until $y = 3$, when the shock hits the fan at $x = 0$. For $y > 3$ the jump condition is

$$x'(y) = \frac{G(u_\ell) - G(u_r)}{u_\ell - u_r} = \frac{G(1 - \sqrt{-x/y}) - G(0)}{1 - \sqrt{-x/y} - 0} = -\frac{1}{3}\frac{(\sqrt{-x/y})^3 - 1}{\sqrt{-x/y} - 1}.$$

2. (a) Since u solves the heat equation, the weak maximum principle certainly applies. Hence the maximum of u over the closed heat cylinder is attained on the parabolic boundary. But $u = 0$ on the spatial boundary, and $u = g \leq 0$ initially. Hence the maximum over the parabolic boundary is 0, so that $u \leq 0$ everywhere by the weak maximum principle.

(b) Because g has compact support in Ω, there is a ball $B(x_0, \epsilon)$ in Ω on which $g \equiv 0$. In particular, at $t = 0$ we see $u = 0$ at all points within distance ϵ of x_0. But $u(x_0, t) < 0$ for all $t > 0$, and so the disturbance (that is, the region on which $u \neq 0$) propagates a distance

ϵ in arbitrarily small time t. Since $\epsilon/t \to \infty$ as $t \to 0$, we conclude that the heat equation allows infinite propagation speed of disturbances.

3. For Duhamel's principle, we first seek $U(x,t;s)$ solving

$$\begin{aligned} U_{tt} &= U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ U(x,0;s) &= 0, \\ U_t(x,0;s) &= e^x. \end{aligned}$$

By D'Alembert's formula, the solution is

$$U(x,t;s) = \frac{1}{2}\int_{x-t}^{x+t} e^{\xi}\, d\xi = e^x \sinh t.$$

(In this case, U is independent of s.) Now Duhamel's principle yields

$$u(x,t) = \int_0^t U(x,t-s;s)\, ds = e^x \int_0^t \sinh(t-s)\, ds = e^x(\cosh t - 1).$$

One can verify directly that this is the desired solution.

4. The condition for weak solutions of $w_{xy} = 0$ is that the following equation be true for all test functions ϕ:

$$\iint_{\mathbb{R}^2} w\phi_{xy}\, dxdy = 0.$$

For u this equation can be written as

$$\iint_{x>0} (2x^2+5)\phi_{xy}\, dxdy = 0$$

and for v it can be written as

$$\iint_{x>0} (x^2+y^2+5)\phi_{xy}\, dxdy = 0.$$

The two integrals are:

$$\text{for } u\text{:} \quad \int_0^\infty \int_{-\infty}^\infty (2x^2+5)\phi_{xy}\, dydx = 0 \qquad \text{for } v\text{:} \quad \int_0^\infty \int_{-\infty}^\infty (x^2+y^2+5)\phi_{xy}\, dydx = 0.$$

Since $\int_{-\infty}^\infty \phi_{xy} dy = \phi_x|_{-\infty}^\infty = 0$ by compact support, the integral for u vanishes for all test functions, and so the solution u is indeed a weak solution.

For the other integral we can certainly use compact support to move the y derivative off ϕ and onto x^2+y^2. Then we can do the x integration. This gives us:

$$\int_0^\infty \int_{-\infty}^\infty (x^2+y^2+5)\phi_{xy}\, dydx = -\int_0^\infty \int_{-\infty}^\infty 2y\phi_x\, dydx = \int_{-\infty}^\infty 2y\phi(0,y)\, dy.$$

Now all you have to do is convince yourself that there is at least one test function ϕ for which the last integral here is non-zero. So v is not a weak solution.

5. (a) $K(x) = -1/4\pi|x|$.

(b) $u(x) = \int_{\mathbb{R}^3} K(x-y)f(y)\,dy$. Note that $K(x-y) \to 0$ as $|x| \to \infty$, uniformly for y in the support of f. Hence $u(x) \to 0$ as $|x| \to \infty$.

(c) The Laplacian is self-adjoint, and so we consider

$$\begin{aligned}
\int u(x)\Delta\phi(x)\,dx &= \int\int K(x-y)f(y)\,dy\,\Delta\phi(x)\,dx \\
&= \int\int K(x-y)\Delta\phi(x)\,dx\,f(y)\,dy \\
&= \int\int \Delta_x K(x-y)\phi(x)\,dx\,f(y)\,dy \\
&= \int\int \delta(x-y)\phi(x)\,dx\,f(y)\,dy \\
&= \int \phi(y)f(y)\,dy \\
&= \int f(x)\phi(x)\,dx.
\end{aligned}$$

(d) Now suppose f is radially symmetric, and is supported in the unit ball. For x outside the unit ball, spherical coordinates give

$$\begin{aligned}
u(x) &= \int_{\mathbb{R}^3} K(x-y)f(y)\,dy \\
&= \int_0^1 \int_{S^2} K(x-\rho x)\,dx\,\rho^2 F(\rho)\,d\rho \\
&= \int_0^1 K(x)\int_{S^2} dx\,\rho^2 F(\rho)\,d\rho
\end{aligned}$$

by the mean value theorem applied to $y \mapsto K(x-y)$, which is harmonic for $|y| < 1$ since $|x| > 1$. Hence

$$u(x) = K(x)\int_{\mathbb{R}^3} f(y)\,dy = K(x).$$

6. —