Math 553 — Fall 2007 — Homework 1

Due: Friday 7 September, by 5pm.

(1) McOwen 1.1.4c
(2) McOwen 1.1.6a
(3) McOwen 1.1.7a
(4) McOwen 1.2.2. Take $u_0 = \frac{1}{2}$ for simplicity. Also, verify the entropy condition.

(5) (See pages 23–24 in McOwen.) Consider the inviscid Burgers' equation

$$\begin{aligned} \left(\frac{1}{2}u^2\right)_x + u_y &= 0 \\ u(x,0) &= k(x) \end{aligned}$$

where $k \in C^1(\mathbb{R})$ has bounded derivative (that is $|k'(x)| \leq A$ for all x, for some constant $A > 0$). Define

$$m := \min_{x \in \mathbb{R}} k'(x).$$

(*Note to Math Grad Students:* technically, this "min" should be an "inf".)

(a) Show that if $m \geq 0$ then the projected characteristics never cross (and so the solution is global, that is, valid for all $y \geq 0$). Draw an illustrative diagram.

(b) Show that if $m < 0$ then the projected characteristics do not cross before time $y = 1/|m|$, but do cross after that time, somewhere. Thus a shock develops someplace either at time $y = 1/|m|$ or arbitrarily soon afterwards. Draw an illustrative diagram.

(*Note to Math Grad Students:* formulate your argument carefully, in part (b).)

(c) These results do not apply when

$$k(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

because this k does not belong to $C^1(\mathbb{R})$. But write $k'(0) = -\infty$ and apply part (b) anyway, then discuss what it tells you about the time of shock formation.