Math 553 — Fall 2007 — Homework 2

Due: Friday 14 September, by 5pm.

(1) McOwen 1.2.3

(2) McOwen 1.2.7. Don't hand this in; just think it over. Note that when McOwen assumes $dq/d\rho = c$, he means that $G(\rho) = c\rho$ in the conservation law, and hence the conservation law reduces to the transport equation as treated in class.

(3) McOwen 1.2.8.

Note the projected characteristics have the form $x = G'(\rho)t + (\text{const})$, as one easily deduces by the method of characteristics. So on this problem, do not use the formula $x = uy + (\text{const})$ (which is valid just for Burgers' equation, where $G'(u) = u$).

Warning. In Burgers' equation, the projected characteristic is a curve along which the *velocity* of the particle is constant, and thus is exactly the path of the particle (in space-time). But for the traffic flow equation, the projected characteristic is a curve along which the *density* of traffic is constant. It is not the path in space-time followed by a vehicle! For further discussion of this point, see the next problem.

(4) We did McOwen 1.2.9 in class. Continuing that problem, suppose a car is initially at a position $x(0) = x_0 < 0$, at time $t = 0$. Determine the position $x(t)$ of this car for all $t > 0$. (Hints: speed=flux/density. Note the car does not move until $t = -x_0/c$.)

(5) For $x \in \mathbf{R}^n, n \geq 2$, write $\vec{u} = \vec{u}(x,t)$ for the velocity vector of some quantity (such as a gas) and $\rho = \rho(x,t)$ for the density, so that the flux vector is $\vec{q} = \vec{q}(x,t) = \rho\vec{u}$. Assume $\vec{u}$ and ρ are smooth functions. Derive the conservation law

$$\rho_t + \nabla \cdot \vec{q} = 0.$$

You can use the Divergence Theorem as stated in the Introduction. Recall the divergence of a vector field is defined by $\text{div}(\vec{q}) = \nabla \cdot \vec{q}$.

HINT: Use Appendix A2c, but explain the key steps in greater detail. Observe, incidentally, that this derivation generalizes the 1-dimensional procedure we used for the traffic flow conservation law, in Sec. 1.2c.

ASIDE: if we made the further assumption that the flux depends on the density as $\vec{q} = \vec{G}(\rho)$, for some vector function $\vec{G}$, then the conservation law would take the form $\rho_t + \nabla \cdot \vec{G}(\rho) = 0$, which nicely parallels our conservation law $u_t + G(u)_x = 0$ in one dimension.

If you get stuck, please email me. Or even better, come and see me at office hours (4-5pm Monday and Friday in 376 Altgeld) or make an appointment to see me at some other time. Best of all, come to the **Homework Study Session**, Wednesday 4-5pm in Altgeld 443.