Math 553 — Fall 2007 — Homework 3

Due: Friday 21 September, by 5pm.

(1) McOwen 2.1.7. *Hint.* See me if you want to be reminded what it means to "separate variables".

Then explain why this example shows illposedness of the Cauchy problem for this PDE, in the upper halfplane. (Consider Cauchy data $u(x,0) = 0, u_y(x,0) = 0$, and call the corresponding solution $u^{(0)}(x,y)$. Then consider Cauchy data $u(x,0) = 0, u_y(x,0) = k^{-1}\sin(kx)$, where $k > 0$ is a constant. Call the corresponding solution $u^{(k)}(x,y)$. Show the Cauchy data for $u^{(k)}$ converges uniformly to the Cauchy data for $u^{(0)}$, as $k \to \infty$. For each x and y, check whether $u^{(k)}(x,y)$ converges to $u^{(0)}(x,y)$ as $k \to \infty$. If this fails for some x and y, then the solution does not depend continuously on the Cauchy data, and so the problem is illposed.)

(2) McOwen 2.2.5.

HINT: the equation is fully nonlinear (not semilinear), so read the bottom half of p. 51 before doing this problem.

REMARK: Monge-type equations are a topic of current research. Notice that the lefthand side of the equation is $u_{xx}u_{yy} - u_{xy}^2$, which is the determinant of the Hessian matrix

$$\text{Hessian} = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}.$$

(3) Let $a, b \in \mathbb{R}$, and define

$$f(x) = \begin{cases} a & \text{if } x < 0 \\ b & \text{if } x > 0 \end{cases}.$$

(a) Show that if $a \neq b$ then f is not weakly differentiable.

Remark. In general, functions with jumps do not have weak derivatives.
Hint. Suppose f does have a weak derivative function, and call it $g(x)$. Use reasoning from class to show $g(x) = 0$ for all $x \neq 0$. Then show this fact contradicts the definition of weak derivative.

(b) Prove f is weakly differentiable in an extended sense, by showing that

$$f' = (b-a)\delta$$

weakly, where δ is the Dirac delta measure concentrated at $x = 0$.

Remark. Part (b) does not contradict part (a), because the Dirac delta is not a function.

(4) Suppose $f(y)$ and $yf(y)$ are both absolutely integrable on $\mathbb{R}$ (that is, they belong to $L^1(\mathbb{R})$). Show that

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} |x-y| f(y)\, dy$$

is a weak solution of $u'' = f$, and that $u(x)$ is continuous.

Remark. This is an example of how a DE can be solved by integrating the nonhomogeneity f against a *fundamental solution*; in this case the DE is $u'' = f$ and the fundamental solution is $\frac{1}{2}|x - y|$. We will return to this idea in McOwen Sec. 2.3c.