Math 553 — Fall 2007 — Homework 4

Due: Friday 28 September, by 5pm.

(1) Fix $p < 1$. Show that

$$\nabla \left(\frac{1}{|x|^p} \right) = \frac{(-p)}{|x|^{p+1}} \frac{x}{|x|} \quad \text{weakly in } \mathbb{R}^2.$$

Hints. This formula is a vector equation with $x = (x_1, x_2) \in \mathbb{R}^2$, and so really you are being asked to prove that each of the two components of the equation holds weakly. In your integrals, use polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$.

Remark. This result shows that functions with certain mild point singularities can still have weak derivatives, in two dimensions. (In one dimension you basically saw last week that a function that is discontinuous at a point does *not* have a weak derivative.)

(2) (Continuous dependence for 1 dim. wave equation) Consider two solutions $u_1(x,t)$ and $u_2(x,t)$ of the wave equation in one dimension, with initial data g_1, h_1 and g_2, h_2 respectively. Prove that

$$\max_x |u_1(x,t) - u_2(x,t)| \le \max_x |g_1(x) - g_2(x)| + t \max_x |h_1(x) - h_2(x)|.$$

(The meaning of this estimate is that if g_1 and g_2 are close together and g_1 and g_2 are close together then the solutions u_1 and u_2 are close together, at time t. That is, the solution depends continuously on the initial data.)

(3) Consider the wave equation in one dimension with initial displacement a triangular shape of width $2c$ and height 2 and initial velocity zero (release from rest). Sketch enough snapshots (graphs of $u(x,t)$ at various times t) to make clear how the solution evolves.

(4) McOwen 3.1.1b.

(5) *Hammer blow problem.* Fix $a > 0$. Consider the wave equation in one dimension with zero initial displacement $u(x,0) = 0$, and with initial velocity $u_t(x,0) = 1$ for $|x| \le a$ and $u_t(x,0) = 0$ for $|x| > a$. (This problem models a piano string that is hit upwards at $t = 0$ by a hammer of width $2a$.)

Sketch snapshots of the string profile $u(x,t)$ at the times $t = a/2c, a/c, 3a/2c, 2a/c, 5a/c$.

Hint. First show that $u(x,t) = \frac{1}{2c}$ (length of $([-a,a] \cap [x-ct, x+ct])$).

(6) McOwen 3.1.4.

Remark. This "reflection" trick is often helpful for PDE problems on half-spaces.