Math 553 — Fall 2007 — Homework 5

Due: Monday 8 October, by noon.

(1) *Hammer blow, continued.* Fix $a > 0$. Consider the one dimensional wave equation on the right half-line $\{x : x > 0\}$, with zero initial displacement $u(x, 0) = 0$, and with initial velocity $u_t(x, 0) = 1$ for $a < x < 3a$ and $u_t(x, 0) = 0$ for all other x. Impose the Neumann boundary condition $u_x(0, t) = 0$ for all $t \geq 0$.

Sketch snapshots of the string profile $u(x, t)$ at the times $t = a/2c, a/c, 2a/c, 5a/c$.

(2) (Continuous L^∞ dependence on the data for the homogeneous wave equation in three dimensions.) Consider the homogeneous wave equation in three dimensions: $u(x, t)$ solves $u_{tt} = c^2 \Delta u$, with initial data $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$. Use analogous notation for solutions u_1 and u_2.

Let $\varepsilon > 0$ and suppose

$$\|g_1 - g_2\|_\infty < \varepsilon, \quad \|\nabla g_1 - \nabla g_2\|_\infty < \varepsilon, \quad \|h_1 - h_2\|_\infty < \varepsilon.$$

Show

$$\|u_1 - u_2\|_\infty < [1 + (c + 1)t]\varepsilon.$$

Notation. $x = (x_1, x_2, x_3)$ and $\|w\|_\infty = \max_x |w(x)|$.

Remark. This problem shows that if g_1 and g_2 are close together, and so are their gradients, and if h_1 and h_2 are close together, then the corresponding solutions u_1 and u_2 are close together. The meaning of "close together" is measured in a pointwise-maximum sense.

(3) *Plane waves in 3 dimensions.* McOwen 3.2.1 parts (a),(b),(c). Then:

(d) Show that for each $\omega \in \mathbf{R}$ and each unit vector $\xi \in S^2$, the function $u(x, t) = e^{i\omega(x \cdot \xi - ct)}$ is a plane wave.

(e) Find a precise statement of the fact that the wave in part (d) moves in the direction of ξ.

Remark. Every plane wave is a linear combination of plane waves having the form $e^{i\omega(x \cdot \xi - ct)}$, as can be seen by representing F in terms of complex exponentials, using the Fourier inversion formula.

Another Remark. Plane waves generalize the left- and right-moving waves, $F(x+ct)$ and $G(x - ct)$, that occur in one dimension.

(4) Consider the nonhomogeneous wave equation in $\mathbf{R}^n$:

$$\begin{cases} u_{tt} - c^2 \Delta u &= f(x, t) \in C^2(\mathbf{R}^n \times [0, \infty)) \\ u(x, 0) &= 0 \\ u_t(x, 0) &= 0 \end{cases}$$

(a) For $n = 1$, derive the formulas

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(\xi, s)\, d\xi ds = \frac{1}{2c} \iint_{\mathcal{T}} f(\xi, s)\, d\xi ds,$$

where $\mathcal{T}$ is the triangle in the xt-plane that is based on the domain of dependence of (x, t) and has its vertex at (x, t).

(b) For $n = 3$, derive the formula

$$u(x,t) = \frac{1}{4\pi c^2} \int_{B^3(ct)} \frac{f(x+y, t-|y|/c)}{|y|}\, dy.$$

Remark. This formula shows that in three dimensions, the response to a driving force decays like the reciprocal of the distance.

(c) Discuss how the answers to (a) and (b) relate to the propagation of sharp signals. *Hint.* Consider a force function f that is supported in a very small region containing the origin, in xt-space. Draw diagrams in xt-space!

(5) EXTRA CREDIT: McOwen 3.2.6. This problem is challenging.