Math 553 — Fall 2007 — Homework 6

Due: Monday 15 October, by noon.

(1) (Continuous L^2 dependence on the data for the homogeneous wave equation in three dimensions.) Consider the homogeneous wave equation in three dimensions: $u(x,t)$ solves $u_{tt} = c^2 \Delta u$, with initial data $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$. Use analogous notation for solutions u_1 and u_2. Assume g_1, g_2, h_1, h_2 have compact support. Let $\varepsilon > 0$ and suppose

$$\|g_1 - g_2\|_2 < \varepsilon, \quad \|\nabla g_1 - \nabla g_2\|_2 < \varepsilon, \quad \|h_1 - h_2\|_2 < \varepsilon.$$

Show

$$\|u_1 - u_2\|_2 < [1 + \sqrt{c^2+1}t]\varepsilon.$$

Hint. Consider

$$\frac{d}{dt}\|u_1 - u_2\|_2^2.$$

Notation. $\|w\|_2 = \left[\int_{\mathbf{R}^3} |w(x)|^2 \, dx\right]^{1/2}$ is called the L^2-norm of the function w.

Remark. This problem shows that if g_1 and g_2 are close together, and so are their gradients, and if h_1 and h_2 are close together, then the corresponding solutions u_1 and u_2 are close together. The meaning of "close together" is measured in the square integrable sense.

(2) (Dispersive equation.) McOwen 3.4.2(b)
Remark. To avoid confusion, change the "c" in the equation to "b". When $b = 0$ this PDE is Airy's equation, which arises in optics.

(3) (Wave equation with lower order terms.)
(i) Suppose

$$v_{tt} - c^2 v_{xx} + \delta^2 v = 0, \qquad -\infty < x < \infty,$$

for some constant $\delta \in \mathbb{R}$. Write

$$w(x,y,t) = \cos(c^{-1}\delta y) v(x,t)$$

and deduce a PDE satisfied by w. (The new PDE will involve x, y and t derivatives.)

(ii) Do similarly for

$$v_{tt} - c^2 v_{xx} - \delta^2 v = 0, \qquad -\infty < x < \infty.$$

Hint. You might want to use the hyperbolic cosine function $\cosh z \overset{\text{def}}{=} (e^z + e^{-z})/2$. Recall also $\sinh z \overset{\text{def}}{=} (e^z - e^{-z})/2$, so that $\cosh' = \sinh$ and $\sinh' = \cosh$.

Conclusion. In class we reduced the one dimensional wave equation with lower order terms to the equations considered in (i) and (ii), with $\gamma = \pm\delta^2$. Then the work above shows how to further reduce to the two dimensional wave equation, which is useful because we already know Hadamard's formula for solving the two dimensional wave equation!

(4) Solve the one dimensional Klein–Gordon equation $u_{tt} - c^2 u_{xx} + c^2 m^2 u = 0$ with initial data g and h. Take $g \equiv 0$ for simplicity.

Hints.

1. When you need to integrate over B^2, you should use Euclidean (rectangular) coordinates.

2. You should end up with the formula (reminiscent of D'Alembert) that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} J_0(ms) h(\xi)\, d\xi$$

where

$$s = \sqrt{c^2 t^2 - (x-\xi)^2}, \qquad J_0(z) = \frac{2}{\pi} \int_0^{\pi/2} \cos(z \sin\theta)\, d\theta.$$

(J_0 is the zero-th Bessel function.)

(5) Read Appendix A.2c in McOwen, on "Ideal fluids and Euler's equations". (Incidentally, Euler was born 300 years ago, in 1707. The Euler equation for an ideal fluid was apparently the second partial differential equation ever written down, after the one dimensional wave equation. You might pause to admire Euler's far-seeing genius.)

Talk to me if you are confused by any point in the discussion. The only work I ask you to hand in is: explain the statement that

Burgers' equation is Euler's equation in one dimension.

You can assume constant pressure.

Extra credit. Explain why the pressure *must* be constant, in order for our description in class of the physical meaning of Burgers' equation to be valid.

(6) (Equipartition of energy) Consider the one dimensional homogeneous wave equation with compactly supported initial data g and h. Show the kinetic and potential energies are equal, for all large time.