Math 553 — Fall 2007 — Homework 7

Due: Monday 29 October, by noon.

(1) (Dirichlet problem for Laplace's equation in the disk.) McOwen 4.1.1. Also, verify directly that $r^k \cos k\theta$ is harmonic.

Hint. Adapt the separation method of Example 2 on p. 105.

(2) (Convexity of mean values.) Let u be subharmonic on a domain Ω in $\mathbf{R}^n, n \geq 3$. Fix $x \in \Omega$.

(a) Show the mean value over spheres is a convex function of $1/r^{n-2}$, meaning

$$\frac{\partial^2}{\partial(1/r^{n-2})^2} \fint_{\partial B(x,r)} u\, dS \geq 0.$$

(Here we restrict to r small enough that $\overline{B(x,r)} \subset \Omega$.) *Hint.* Use a Remark from class, or Darboux.

(b) State without proof an analogous result for subharmonic functions in *two* dimensions. *Hint.* What is the harmonic function in two dimensions that is analogous to $1/r^{n-2} = 1/|x|^{n-2}$?

(3) Assume u is harmonic in $\mathbb{R}^n$ and $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and convex ($\phi'' \geq 0$). Prove $v = \phi(u)$ is subharmonic. *Hint.* Jensen's inequality, or direct computation.

Examples. $\phi(u) = u^2, \phi(u) = e^u$.

(4) Prove the Laplacian is rotationally invariant; that is, show that if O is an orthogonal $n \times n$ matrix then $\Delta(u(Ox)) = (\Delta u)(Ox)$.

Hence in particular, if u is harmonic then so is the rotated function $u(Ox)$.

Remark. The Laplacian is defined to be the sum of the pure second order partial derivatives with respect to the *rectangular* coordinates $x_1, \ldots, x_n$, and this definition does not suggest the Laplacian is rotationally invariant. But it is!

(5) Compute the potential $u(x)$ due to a solid unit ball of charge in $\mathbb{R}^3$, with charge density $\rho = 1$ for $|x| \leq 1$ and $\rho = 0$ for $|x| > 1$, and with $u(\infty) = 0$. Note there is at most one continuous weak solution to this problem (we haven't quite proved this, but it essentially follows from the uniqueness arguments in Section 4.1).

Hints. Look for a solution $u(x)$ that is radial, continuous, and smooth except at $|x| = 1$. [Explain why you would *expect* a solution of the problem to have these properties.] Thus u satisfies Laplace's equation classically outside the ball and Poisson's equation classically inside it. You should also require the first derivatives of u to be continuous at $|x| = 1$. Then you should be able to show $\Delta u = -4\pi\rho$ weakly.

The logic in this problem is: *if* you can find a solution satisfying the above conditions, *then* you've solved the problem.