Math 553 — Fall 2007 — Homework 8

Due: Monday 5 November, by noon.

(1) (Kelvin inversion) Prove that if $u(x)$ is harmonic in $\Omega \subset \mathbb{R}^n, n \geq 2$, then

$$v(x) = \frac{1}{|x|^{n-2}} u(x^*)$$

is harmonic in $\Omega^* \stackrel{\text{def}}{=} \{y^* : y \in \Omega\}$, where we define $x^* = x/|x|^2$ to be the reflection of x in the unit sphere. *Hint.* Compute Δv.

Note 1. Check for yourself that $(x^*)^* = x$ and that $0^* = \infty, \infty^* = 0$. Thus the Kelvin inversion turns problems for Laplace's equation on unbounded domains into problems on bounded domains.

Note 2. The function v is called the *Kelvin inversion* of u. Observe that in two dimensions, the Kelvin inversion of $u(z)$ is simply $u(1/\overline{z})$ where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C} \simeq \mathbb{R}^2$, because you can check that $z^* = 1/\overline{z}$.

(2) Show the Green function for the unit disk in $\mathbb{R}^2$ can be written in the simple form

$$G(z, w) = \frac{1}{2\pi} \log \left| \frac{z - w}{1 - z\overline{w}} \right|,$$

where we regard z and w as complex numbers (in other words, writing the vector (x_1, x_2) as a complex number $z = x_1 + ix_2$ and so on).

(3) McOwen 4.2.1(a)

Remark. This gives a simple formula for the Poisson kernel of the unit disk in $\mathbb{R}^2$.

(4) (Negativity of the Green function.) Show $G(x, y) < 0$ for all $x, y \in \Omega, x \neq y$.

Notation. Many authors call $-G$ the Green function, so that their Green function is positive.

(5) (Symmetry of the Green function.) Let Ω be a bounded smooth domain in $\mathbb{R}^n, n \geq 2$. Assume that for each $x \in \Omega$ a "corrector" function exists, that is, a harmonic function $w_x(y)$ with $w_x \in C^2(\overline{\Omega})$ and $w_x(y) = -K(x-y)$ for $y \in \partial\Omega$. Define the Green function

$$G(x, y) = K(x - y) + w_x(y),$$

so that $G(x, y) = 0$ when $x \in \Omega, y \in \partial\Omega$.

Prove the Green function is symmetric: $G(x, y) = G(y, x)$ whenever $x, y \in \Omega, x \neq y$.

Hints. Fix $x, y \in \Omega, x \neq y$, and define $u(z) = G(x, z)$ and $v(z) = G(y, z)$ for $z \in \Omega$, so that $u \in C^2(\overline{\Omega} \setminus \{x\}), v \in C^2(\overline{\Omega} \setminus \{y\})$. Use Green's second formula on the domain formed from Ω by removing balls of radius ϵ centered at x and y.