

Math 444 — Spring 2001 — Test

Total points: 100. Do all questions.

Instructions: Show ALL your working and make your explanations as full as possible, unless the question says otherwise. Calculators are not allowed on this exam; neither are books or notes.

If you apply a theorem from class, you need not state the theorem but you **must** demonstrate that its hypotheses are satisfied.

Green's Formulas:

$$\begin{aligned}\int_{\Omega} [v\Delta u + \nabla v \cdot \nabla u] dx &= \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS \\ \int_{\Omega} [v\Delta u - u\Delta v] dx &= \int_{\partial\Omega} \left[v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right] dS\end{aligned}$$

1: (25 points) Assume G is a smooth function and $u(x, y)$ is a weak solution of the conservation law

$$G(u)_x + u_y = 0.$$

(a) Show that if u is in fact a *smooth* solution except for a jump across the C^1 -curve $x = \xi(y)$, then the direction of the curve is related to the jump by

$$\xi'(y) = \frac{G(u_\ell) - G(u_r)}{u_\ell - u_r}.$$

(b) State the entropy condition, say what it means physically, and illustrate with a diagram. (If you like, you can assume in part (b) that $G(z) = \frac{1}{2}z^2$, *i.e.* Burgers' equation.)

2: (25 points) Let $f(x, t)$ be smooth.

(a) State Duhamel's Principle for solving the nonhomogeneous wave equation $z_{tt} - c^2\Delta z = f$ in \mathbb{R}^n (with zero initial conditions) in terms of the solutions to certain homogeneous problems. (You are not required to solve these homogeneous problems.)

(b) Prove Duhamel's Principle. That is, show that your formula for $z(x, t)$ really does solve $z_{tt} - c^2\Delta z = f$.

3: (25 points) For $x \in \mathbb{R}^2$, let

$$u(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \leq 1, \\ \ln(|x|^2) & \text{if } |x| > 1, \end{cases} \quad f(x) = \begin{cases} 4 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Show $\Delta u = f$ weakly in \mathbb{R}^2 . (Note: here $x = (x_1, x_2)$ and $|x| = \sqrt{x_1^2 + x_2^2}$.)

4: (25 points) Consider the nonhomogeneous wave equation $u_{tt} - c^2 \Delta u = f$. Write

$$E(t) = \frac{1}{2} \int (u_t^2 + c^2 |\nabla u|^2) dx.$$

(Here the integral is over \mathbb{R}^n . We assume u and f are smooth and have compact support, at each t .)

(a) Show $E'(t) = \int u_t f dx$.

(b) Explain why this formula is physically plausible.

[For part (b) you can work in one dimension, so that f represents an external force on the string.]