Math 444 — Spring 2001 — Test Solutions

Green's Formulas:

$$\int_\Omega [v\Delta u + \nabla v \cdot \nabla u]\, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, dS$$

$$\int_\Omega [v\Delta u - u\Delta v]\, dx = \int_{\partial\Omega} \left[v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right] dS$$

1: (25 points) Assume G is a smooth function and $u(x,y)$ is a weak solution of the conservation law

$$G(u)_x + u_y = 0.$$

(a) Show that if u is in fact a *smooth* solution except for a jump across the C^1-curve $x = \xi(y)$, then the direction of the curve is related to the jump by

$$\xi'(y) = \frac{G(u_\ell) - G(u_r)}{u_\ell - u_r}.$$

(b) State the entropy condition, say what it means physically, and illustrate with a diagram. (If you like, you can assume in part (b) that $G(z) = \frac{1}{2}z^2$, *i.e.* Burgers' equation.)

Solution:

(a) Write $u_\ell = u(\xi(y)-, y), u_r = u(\xi(y)+, y)$. Then $u_\ell \neq u_r$ because of the jump. Observe that

u is a weak solution

$$\Longrightarrow \quad 0 = G(u(b,y)) - G(u(a,y)) + \frac{d}{dy}\left\{ \int_a^{\xi(y)} u(x,y)\, dx + \int_{\xi(y)}^b u(x,y)\, dx \right\}$$

whenever $a < \xi(y) < b$

$$\Longrightarrow \quad 0 = G(u(b,y)) - G(u(a,y)) + \int_a^{\xi(y)} u_y(x,y)\, dx + u_\ell \xi'(y) + \int_{\xi(y)}^b u_y(x,y)\, dx - u_r\xi'(y)$$

by differentiating through the integral, and also using the fundamental theorem to differentiate the limit of integration, $\xi(y)$

$$\Longrightarrow \quad 0 = -G(u_\ell) + G(u_r) + \xi'(y)\,[u_\ell - u_r]$$

by substituting $u_y = -G(u)_x$ in the two integrals (which is valid classically away from $x = \xi(y)$)

$$\Longrightarrow \quad \xi'(y) = \frac{G(u_\ell) - G(u_r)}{u_\ell - u_r}$$

(b) The entropy condition is: $G'(u_\ell) > \xi'(y) > G'(u_r)$. Physically it means that two characteristics can "come together" or collide at a shock, but cannot emanate from the shock and then subsequently move apart. We drew some diagrams of this in class.

2: (25 points) Let $f(x,t)$ be smooth.
(a) State Duhamel's Principle for solving the nonhomogeneous wave equation $z_{tt} - c^2\Delta z = f$ in $\mathbb{R}^n$ (with zero initial conditions) in terms of the solutions to certain homogeneous problems. (You are not required to solve these homogeneous problems.)
(b) Prove Duhamel's Principle. That is, show that your formula for $z(x,t)$ really does solve $z_{tt} - c^2\Delta z = f$.

Solution:

(a) Duhamel's Principle states that if $Z(x,t;s)$ solves the homogeneous wave equation $Z_{tt} - c^2\Delta Z = 0$ with initial conditions $Z(x,0;s) = 0$ and $Z_t(x,0;s) = f(x,s)$, then

$$z(x,t) = \int_0^t Z(x, t-s; s)\, ds$$

solves the nonhomogeneous wave equation $z_{tt} - c^2\Delta z = f$ in $\mathbb{R}^n$ with zero initial conditions.

(b) The proof of Duhamel's Principle for dimension one was given in class, and can be found on page 81 of MCOWEN. For higher dimensions, just replace Z_{xx} with ΔZ, and so on.

3: (25 points) For $x \in \mathbb{R}^2$, let

$$u(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \le 1, \\ \ln(|x|^2) & \text{if } |x| > 1, \end{cases} \qquad f(x) = \begin{cases} 4 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Show $\Delta u = f$ weakly in $\mathbb{R}^2$. (Note: here $x = (x_1, x_2)$ and $|x| = \sqrt{x_1^2 + x_2^2}$.)

Solution: Write $B = \{x : |x| < 1\}, C = \{x : |x| > 1\}$. Then $\Delta u = 4$ in B, since $u = x_1^2 + x_2^2 - 1$ there. And $\Delta u = 0$ in C since $\ln |x|$ is harmonic in $\mathbb{R}^2$ except at the origin. Thus $\Delta u = f$ pointwise in B and in C.

For all $v \in C_0^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} u \Delta v \, dx$$

$$= \int_B u \Delta v \, dx + \int_C u \Delta v \, dx$$

$$= \left(\int_B v \Delta u \, dx + \int_{\partial B} u \frac{\partial v}{\partial \nu} \, dS - \int_{\partial B} v \frac{\partial u}{\partial \nu} \, dS \right) + \left(\int_C v \Delta u \, dx + \int_{\partial C} u \frac{\partial v}{\partial \nu} \, dS - \int_{\partial C} v \frac{\partial u}{\partial \nu} \, dS \right)$$

by Green's Second Formula; because v has compact support we can ignore the "boundary" terms near infinity

$$= \left(\int_B v f \, dx - \int_{\partial B} v \frac{\partial u}{\partial \nu} \, dS \right) + \left(\int_C v f \, dx - \int_{\partial C} v \frac{\partial u}{\partial \nu} \, dS \right)$$

using that $\Delta u = f$ pointwise in B and C, and that $u = 0$ on ∂B and ∂C (where $|x| = 1$)

$$= \int_{\mathbb{R}^2} v f \, dx$$

because $\partial B = \partial C$, and $(\partial u / \partial \nu) = u_r = 2|x| = 2$ on ∂B

while $(\partial u / \partial \nu) = -u_r = -2/|x| = -2$ on ∂C.

Hence $\Delta u = f$ weakly in $\mathbb{R}^2$.

Comment: If you understand Green's Formulas, you will go a long way!

4: (25 points) Consider the nonhomogeneous wave equation $u_{tt} - c^2 \Delta u = f$. Write

$$E(t) = \frac{1}{2} \int \left(u_t^2 + c^2 |\nabla u|^2 \right) dx.$$

(Here the integral is over $\mathbb{R}^n$. We assume u and f are smooth and have compact support, at each t).

(a) Show $E'(t) = \int u_t f \, dx$.

(b) Explain why this formula is physically plausible.

[For part (b) you can work in one dimension, so that f represents an external force on the string.]

Solution: (a)

$$\begin{aligned}
& E'(t) \\
= \ & \frac{d}{dt} \frac{1}{2} \int \left(u_t^2 + c^2 \nabla u \cdot \nabla u \right) \, dx \\
= \ & \int \left(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \right) \, dx \qquad \text{(differentiation through the integral is valid since} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\quad u_t \text{ and } \nabla u \text{ have compact support)} \\
= \ & \int \left(u_t \left[c^2 \Delta u + f \right] + c^2 \nabla u \cdot \nabla u_t \right) \, dx \qquad \text{by the PDE} \\
= \ & \int \left(c^2 \nabla \cdot [u_t \nabla u] + u_t f \right) \, dx \\
= \ & \int u_t f \, dx \quad \text{by the divergence theorem, since } \nabla u \equiv 0 \text{ near infinity.}
\end{aligned}$$

[Note: Another way to organize this calculation is to just multiply the PDE by u_t and integrate with respect to x.]

(b) The formula $E'(t) = \int u_t f \, dx$ is physically plausible as follows. For example, if $f > 0$ and $u_t > 0$ then the force and velocity (of the string) are both upwards, so that the force will increase the velocity (and hence the kinetic energy) beyond what it would have been otherwise. Since the total energy is conserved in the absence of any external force, we see that the presence of f leads to an increase in the total energy. This is confirmed by the formula: if $f > 0$ and $u_t > 0$ then $E'(t) = \int u_t f \, dx > 0$.