

Math 561 — Spring 2006 — Final Exam Guidelines

The exam will be 3 hours long, on Wednesday 10 May, 1:30–4:30pm, in the usual classroom.

Material: everything covered in class and homework, except new material in the final two days of class. Only topics covered in class are examinable; you are not responsible for sections of the text that we didn't cover in class.

The exam will be approximately 1/3 on material covered up to the test, and approximately 2/3 on material covered since the test.

Read again the advice in the handout “How to Prosper in Math Graduate School”. Especially notice the part about thoroughly understanding every **homework** problem. And learn the **definitions** and the statements of all major **theorems** and **formulas**. Memorize the main probability densities that we use as examples (Bernoulli, Binomial, Geometric, Poisson, Uniform, Normal, Exponential).

One of the problems on the test might ask you to reproduce a proof. Following are the proofs you should learn. (You are not required to learn the statements of the results, although it is helpful to do so.)

- Theorem 1.4.2 (independent π -systems generate independent σ -algebras). For notational simplicity, just learn the special case $n = 2$.
- Theorem 1.5.2 (L^2 Weak Law of Large Numbers for i.i.d.s), including the proof of Lemma 1.5.3 (Convergence in L^p implies convergence in probability).
- First Borel–Cantelli Lemma 1.6.1.
- Theorem 1.6.5 (L^4 Strong Law of Large Numbers).
- Second Borel–Cantelli Lemma 1.6.6.
- Wald's equation 3.1.6.
- Theorem 3.2.3 for dimensions 2 and 3 (recurrence of simple random walk).
- Examples 4.1.1, 4.1.2, 4.1.3. (Note Example 4.1.3 shows $E(X|\mathcal{F})$ is a piecewise constant r.v. with $E(X|\mathcal{F}) = E(X|\Omega_i)$ on Ω_i .)
- Theorem 4.1.2 (the Tower Property).
- Theorem 4.1.4 (if $\mathcal{F} \subset \mathcal{G}$ then $E(X|\mathcal{F})$ is the best approximation in $L^2(\mathcal{F})$ to $X \in L^2(\mathcal{G})$).
- Example 4.2.1, and the related example in class using products of i.i.d.s rather than sums.
- Theorem 4.2.7 (predictable strategies take supermartingales to supermartingales).
- Corollary 4.2.8 (stopping times take supermartingales to supermartingales).
- Theorem 4.4.5 (L^p convergence theorem).
- Section 4.7, Stopping Theorem parts (ii) and (iii) (for submartingales bounded above, and submartingales with bounded increments).

Consult your notes, because in some cases the proof in class was more detailed than the one in the textbook.

There are many smaller topics you should learn and understand as well. Re-read *all* your lecture notes carefully.

And here are a few exercises to try from the textbook, covering the last part of the course:

- Exercise 4.4.2, page 247.
- Exercise 4.4.8, page 251. *Hint.* You can use Theorem 4.4.6 (which is just a special case of the second formula on page 226).

Aside. For interpreting this result, note the righthand side is the sum of the covariances of the increments. Thus the result says roughly that “if the increments of the martingales are positively correlated, then $E(X_n Y_n)$ will increase with n ”.

You could imagine tossing two fair coins repeatedly, where you win \$1 for heads and lose \$1 for tails, on each coin. Write X_n and Y_n for your fortune due to the two coins, respectively. The Exercise implies that if the two coins are independent then $E X_n Y_n = E X_0 Y_0$, whereas if the coins are consistently correlated (say, always giving the same result), then $E X_n Y_n \neq E X_0 Y_0$.

- Exercise 4.5.3, page 260. *Remark.* Consequently, f is differentiable a.s., with $X_\infty = f'$ a.s.
- Exercise 4.6.2, page 263. *Hint.* Adapt the proof of Theorem 7.5.9. (Note N in that proof is just a number, not a stopping time.)

Remark. This result is a backwards conditional dominated convergence theorem with decreasing σ -fields.

Plan your study schedule today, so that you will have time to learn everything!