Math 561 — Spring 2006 — Practice Exercises #1 (not for handing in)

Exercise 1.1.10. Page 8. The last line has a typo: it should say "When $g(x) = ax+b$ with $a > 0$, the answer is $f((y-b)/a)/a$."

Exercise 1.2.1, page 9, rephrased. Engineering students can omit this problem. Let $(\Omega_i, \mathcal{F}_i)$ be measurable spaces, for $i = 1, 2$, and suppose $X : \Omega_1 \to \Omega_2$ is a measurable function. Write $\sigma(X) = X^{-1}(\mathcal{F}_2) = \{X^{-1}(B) : B \in \mathcal{F}_2\}$ for the σ-field on Ω_1 generated by X. (It is easy to check $\sigma(X)$ really is a σ-field.)

Show that if $\mathcal{B} \subset \mathcal{F}_2$ generates $\mathcal{F}_2$ then $X^{-1}(\mathcal{B})$ generates $X^{-1}(\mathcal{F}_2) = \sigma(X)$.

[*Extended Hint:* Let $\mathcal{A}$ be any σ-field containing $X^{-1}(\mathcal{B})$. Let $\mathcal{C} = \{B : X^{-1}(B) \in \mathcal{A}\}$. Show $\mathcal{C}$ is a σ-field containing $\mathcal{B}$, hence $\sigma(\mathcal{C})$ contains $\mathcal{F}_2$. Deduce $X^{-1}(\mathcal{F}_2) \subset \mathcal{A}$. Thus $X^{-1}(\mathcal{F}_2) \subset \sigma(X^{-1}(\mathcal{B}))$. The converse inclusion is easy, completing the proof.]

Exercise 1.3.8. Page 15.

Exercise 1.3.13. Page 21.

Easy exercise used in proving Corollary 1.4.5. Show that if random variables X_i are $\mathcal{G}_i$-measurable for $i = 1, \ldots, n$ (where $\mathcal{G}_i \subset \mathcal{F}$ for all i) and the $\mathcal{G}_i$ are independent σ-fields, then the X_i are independent random variables.

Exercise 1.4.5. Page 25.

Exercise 1.4.6. Page 28.

You're welcome to consult me about these problems during **office hours**: Tuesday 5-6pm, Friday 2-3pm, and other times by e-mail appointment.

Math 561, Spring 2006 — Homework 1

Due by **5pm Friday 17 February**, to Altgeld 376.

- Exercise 1.4.6, page 28.
- Exercise 1.5.1, page 45.
(This is an L^2 Weak Law with unbounded variances that grow sublinearly.)
- Exercise 1.5.4, page 45.
- Use Exercise 1.5.8, page 45, to answer the following. Consider X_k to be your winnings on the kth play of a game, and write a paragraph answering the questions: "In what respect is the game *fair*? In what respect is it *unfair*?"

  Your answer should deal with the facts that $ES_n = 0$ and $S_n/n \to 0$ in probability (why?) and $S_n/(n/\log_2 n) \to -1$ in probability.

Remark relevant to the last problem. A gambler cares about the probability distribution of the total winnings S_n, not so much about the average winnings S_n/n. The two are equivalent as far as expected value is concerned (due to linearity of expectation), but convergence in probability is a different matter. For instance if the Weak Law tells us that $S_n/n - \mu \to 0$ in probability, then we know for all $\varepsilon > 0$ that

$$P(n(\mu - \varepsilon) \le S_n \le n(\mu + \varepsilon)) \to 1 \qquad \text{as } n \to \infty,$$

which is definitely a weaker statement than

$$P(n\mu - \varepsilon \le S_n \le n\mu + \varepsilon) \to 1 \qquad \text{as } n \to \infty,$$

which would mean $S_n - n\mu \to 0$ in probability. The gambler might *want* the last statement to be true, but that doesn't make it true...

Math 561, Spring 2006 — Homework 2

Due by **5pm Monday 27 February**, to Altgeld 376.

- Exercise 1.6.8, page 51. *Hints.* You can use Exercise 1.4.16 (page 33), and facts about the Poisson distribution from the first-day handout. The proof of Theorem 6.8 (page 50) provides useful guidance.

- Exercise 1.6.10, page 54.

- Exercise 1.6.17, page 54. *Hint.* Let $K > 0$ be arbitrary and show that almost surely, $X_n > Kn \log_2 n$ i.o.

- Exercise 1.6.18, page 54. *Aside.* These are exponential random variables, which measure waiting times for events that happen at a rate of one per unit time on average (like bus arrivals on Wright Street). It is a good exercise to say in words what part (ii) of this problem is telling us, about such waiting times.

- 1.7.2, page 58.

- 1.7.3, page 60.

Math 561, Spring 2006 — Homework 3

Due by **5pm Monday 6 March**, to Altgeld 376.

- Exercise 1.7.4, page 60. In part (i), you need to find a formula for the limiting value $c(p)$. In part (ii), just differentiate formally through the expectation (*i.e.* rigorous justification is optional).

  Notes. The bonds and stocks are worth \$1 each, at the beginning of a year. At the end of the year, the bonds are worth \$a and the stocks are worth \$V. The constant a is assumed positive.

- Exercise 2.1.3, pages 79 and 80.

  Hint. First show $\frac{1}{2n} \log P(S_{2n} = 2k) \to -\gamma(a)$ as $n \to \infty$, assuming $k/n \to a$. Then show $P(S_{2n} \geq 2k)$ is comparable to $P(S_{2n} = 2k)$.

  Note. There is a different proof of this "large deviations" estimate on pages 74–75, following the method we used in class.

- (*Extra credit.*) Exercise 1.9.6, page 75. Recall the moment generating function for a fair coin toss X is $\phi(\theta) = E(e^{\theta X}) = (e^{\theta} - e^{-\theta})/2$.

  Remark. This problem gives a simple, explicit estimate on the probability of getting a lot more heads than tails (by choosing $0 < a < 1$).

- Exercise 2.2.2(iii), page 83.

  Remark. You studied the random variable with different tools (Borel–Cantelli) in Exercise 1.6.18, page 54.

- Exercise 2.2.8, page 88.

- Exercise 2.2.9, page 89.

- Exercise 2.2.10, page 89.

Math 561, Spring 2006 — Homework 4

Due by **5pm Wednesday 29 March**, to Altgeld 376.

- Exercise 2.2.11, page 89.
- Exercise 2.3.1, page 93.
- Exercise 2.3.4, page 95.
  Remark. Earlier in the course we pointed out that the sum of independent normal random variables is normal, and that this can be proved by convolution of the distribution functions. Here you should instead argue more briefly, using characteristic functions.
- Exercise 2.3.17, page 101.
- Exercise 2.4.4, page 114.
- Exercise 2.4.5, page 114.
- Exercise 2.4.10, page 119. Note $X_1, X_2, \ldots$ are assumed independent.
  Remark. This is a significant improvement over the Central Limit Theorem, for uniformly bounded random variables, because here we do not assume identical distribution.
- Exercise 2.4.11, page 119. Note $X_1, X_2, \ldots$ are assumed independent.
  Remark. This is a significant improvement over the Central Limit Theorem, for random variables with uniformly bounded $(2+\delta)$-moments, because here we do not assume identical distribution, just identical variances.
- Exercise 2.4.12, page 119.

Math 561, Spring 2006 — Homework 5

Due by **5pm Friday 7 April**, to Altgeld 376.

- Exercise 3.1.1, page 173.
- Exercise 3.1.4, page 174.
- Exercise 3.1.8, page 177. And explain in words what parts (i) and (ii) of the problem are telling us.
- Exercise 3.1.12, page 179. Note $e/2 \approx 1.36$, which is quite a bit larger than 1.
- Exercise 3.1.14, page 180. (Here $a \in \mathbb{R}$ is fixed.)

  Hints. (i) First compute $P(T = k), k = 1, 2, 3, \ldots$. (ii) The problem means: take $a = \alpha$, and show that $EY_\tau \leq \alpha = EY_T$. Thus the optimal stopping strategy, in order to maximize the expected value of Y_n at the stopping time, is to use the stopping criterion T with $a = \alpha$.

  Note. I expect you to prove the displayed inequality, in the statement of part (ii).

  Remark. You can interpret Y_n as your winnings in a game where you pay $\$cn$ for n "attempts" to win, but where you only get to keep the winnings from your most successful attempt. Or imagine a hunter who expends c calories for each animal he catches (with the mth animal providing $X_m \geq 0$ calories), but such that the hunter can only carry one animal home to eat (the biggest). What is the significance of having $\alpha \geq 0$, for the hunter?!

Math 561, Spring 2006 — Homework 6

Due by **5pm Friday 14 April**, to Altgeld 376.

- (*Not for handing in.*) (*Bayes' Formula*) Exercise 4.1.2, page 220. Note in the second part of the problem, you are taking $\mathcal{G} = \sigma(G_1, G_2, \ldots)$ where the G_j partition Ω.

- (*Not for handing in.*) People come in 10 types, with a randomly chosen person having probability $\frac{j}{55}$ of being type j, for $j = 1, 2, \ldots, 10$. The probability that a person of type j likes wine is $\frac{j-1}{45}$.

  Given that a person likes wine, what is the probability that the person is of type 3? *Answer:* 1/55.

- Exercise 4.1.5, page 224. *Hint.* Be careful about exceptional sets of probability zero.

- Exercise 4.1.8, page 226. Conclude that $\mathcal{G} \subset \mathcal{F}$ implies
$$E(X - E(X|\mathcal{G}))^2 \geq E(X - E(X|\mathcal{F}))^2$$
which says that the more information you have in your σ-field, the closer the conditional expectation gets to the random variable.

- Exercise 4.1.9, page 226. Conclude that conditioning preserves the mean and reduces the variance:
$$EX = E(E(X|\mathcal{F})), \qquad \mathrm{var}(X) \geq \mathrm{var}(E(X|\mathcal{F})).$$

- Exercise 4.1.10, page 226. Also show $EX = \mu EN$.

  Remark. This problem gives the mean and variance for summing a random number of random numbers.

- Exercise 4.1.11, page 226.

Math 561, Spring 2006 — Homework 7

Due by **5pm Monday 24 April**, to Altgeld 376.

- Exercise 4.2.13, page 235.

- Exercise 4.3.1, page 237.

- Exercise 4.3.4, page 237.
  Hint. You need to create a supermartingale and then stop it at $N = \inf\{k : \sum_{m=1}^{k} Y_m > M\}$.
  Aside. When $Y_n \equiv 0$, we see that X_n is a nonnegative supermartingale, in which case this exercise is a special case of Corollary 2.11 on page 233.

- Exercise 4.1.1, page 218. Then show that if $X_1, X_2, \dots$ are i.i.d. with mean $\mu = EX_i$ and if N is an independent integer-valued random variable, then
$$E(X_1 + \cdots + X_N | N) = \mu N.$$

- *Branching process.* Consider the branching process studied in class (Section 4.3d). Define
$$\begin{aligned} \phi(s) &= \sum_{k=0}^{\infty} P(\xi = k) s^k, && 0 \le s \le 1, \\ &= E(s^{\xi}), && 0 < s \le 1, \end{aligned}$$
and similarly for $n \ge 1$ let
$$\begin{aligned} \phi_n(s) &= \sum_{k=0}^{\infty} P(Z_n = k) s^k, && 0 \le s \le 1, \\ &= E(s^{Z_n}), && 0 < s \le 1. \end{aligned}$$
Aside. These are rescaled moment generating functions: $\phi(s) = E(e^{\xi \log s})$.
  (a) Show $\phi_n = \phi \circ \cdots \circ \phi$ (an n-fold composition).
  Hint. Induction. Calculate $E(s^{Z_{n+1}} | Z_n)$.
  (b) Deduce $\theta_n := P(Z_n = 0) = (\phi \circ \cdots \circ \phi)(0) = \phi(\theta_{n-1})$.
  Remark. This is a reformulation of page 245 statement (a).
  (c) Read pages 245–246 statements (b), (c), and then do Exercise 4.3.13 on page 246.
  Note. When reading the proof of statement (b), it helps to remember that the function $\phi(s) - s$ is strictly convex, and is nonnegative at $s = 0$ and is zero at $s = 1$, and has positive slope (derivative) at $s = 1$. The existence of a unique point $\rho \in (0, 1)$ at which $\phi(\rho) - \rho = 0$ now follows easily from the intermediate value theorem.
  Note that the proof of statement (b) and the fact that ϕ is increasing imply
$$x \le \phi(x) < \phi(\rho) = \rho, \qquad \forall\, x \in [0, \rho).$$
This helps when reading the proof of statement (c), because it shows $\theta_{n-1} \le \phi(\theta_{n-1}) = \theta_n < \rho$.

Math 561, Spring 2006 — In-class work

- Exercise 4.2.2, page 229 (slightly adapted): Let $x_0 \in \mathbb{R}^d$, and suppose $\xi_1, \xi_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^d$ with ξ_i taking values uniformly distributed on the unit sphere (so in particular $|\xi_i| = 1$). Define $S_0 = x_0$ and $S_n = S_{n-1} + \xi_n$ for $n \geq 1$.

  Show that if f is superharmonic on $\mathbb{R}^d$, then $f(S_n)$ is a supermartingale with respect to the natural filtration $\mathcal{F}_0 = \{\phi, \Omega\}, \mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. You may assume $f(S_n) \in L^1$.

  Hint. Recall that a (lower semicontinuous) function is called *superharmonic* if it satisfies the super-mean value property:
$$f(y) \geq \frac{1}{\text{Area}(S)} \int_S f(y+z)\, dS(z)$$
  for every sphere S centered at the origin and every $y \in \mathbb{R}^d$.

  Remark. This explains why Doob named supermartingales "super"!

  Asides. When f is twice continuously differentiable, superharmonicity is equivalent to $\Delta f = \nabla^2 f \leq 0$ (see Math 553 Partial Differential Equations). Thus in one dimension, superharmonicity is just concavity: $f'' \leq 0$.

  In three dimensions the function $f(x) = 1/|x|$ is superharmonic (as proved in Math 553), and this has physical relevance since $1/|x|$ is the potential due to a point mass or charge. Incidentally, $f(x) = 1/|x|$ is *harmonic* away from the origin, meaning the mean value property holds with equality so long as the origin is outside the sphere $y + S$.

- Exercise 4.7.1, page 271: Show that if X_n is a nonnegative supermartingale then $\lambda P(\sup_n X_n > \lambda) \leq EX_0$, for each $\lambda > 0$.

- *Stochastic control.* [D. Williams, "Probability with Martingales" page 234.] The control system on the Starship *Enterprise* has gone wonky: all Scotty can do is decide on a distance to be travelled, and then the ship will "hop" that distance in a randomly chosen direction, and stop. Then Scotty can repeat the process. The ship clearly needs to return for repairs to the Solar System, which is the open ball of radius r centered at the origin.

  Initially the *Enterprise* is at distance R_0 from the origin (where $R_0 > r$). Let R_n be its distance from the origin after n "space hops" of the kind described above.

  (a) Show that whatever strategy is used for choosing the distances to be travelled in these hops, the reciprocal distance $1/R_n$ is a supermartingale. In particular $E(1/R_n) \leq 1/R_0$.

  (b) Suppose your strategy always chooses the distance of the next hop to be less than the current distance to the Sun. Show that $1/R_n$ is a martingale.

  (c) Captain Kirk needs to know the probability p that the *Enterprise* will eventually reach the Solar System. Show that for every hopping strategy, $p \leq r/R_0$.

  (d) *Optimal stochastic control.* Spock says it is logical that given any $\varepsilon > 0$, there exists a strategy for choosing the hop distances so that $p > (r/R_0) - \varepsilon$. Discuss.

Math 561, Spring 2006 — More in-class work

- Problem A. Is a simple random walk in $\mathbb{Z}^d, d \geq 1$, uniformly integrable? Explain.

- Problem B. Let X_n be the simple random walk in one dimension, and let N be the stopping time for hitting 1 (that is, $N = \inf\{n > 0 : X_n = 1\}$. Show $X_{N \wedge n}$ is a martingale that converges a.s. but not in L^1.

- Problem C. A friend has a great "system", consisting of a betting strategy and a stopping criterion, that is guaranteed to win money. (To learn the basic rules of roulette, see the *Introductory Lecture* in Durrett's book.)

  What can you predict about the behavior of your friend's fortune X_n, when playing with the system? (You may assume the friend starts with $X_0 = 0$ dollars.)