Math 561 — Spring 2006 — Test Solutions

Total points: 100. 75 minutes. Show ALL your working and make your explanations as full as possible. Electronic devices are not allowed; neither are books or notes.

1: (30 points) **Do part (a) or part (b), but not both.**

(a) Prove that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are π-systems that are independent, then $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.

Solution. Durrett Theorem 1.4.2 (page 24), with $n = 2$.

Your solution should clearly point out where the independence hypothesis on $\mathcal{A}_1$ and $\mathcal{A}_2$ is used, namely to show that $\mathcal{A}_1 \subset \mathcal{L}$.

(b) (L^4 Strong Law) Let $X_1, X_2, \dots$ be i.i.d. with $EX_i = \mu$ and $EX_i^4 < \infty$. Write $S_n = X_1 + \cdots + X_n$. Prove $S_n/n \to \mu$ a.s.

Solution. Durrett Theorem 1.6.5 (page 48).

Your solution needs to clearly show where the independence of $X_1, X_2, \dots$ is used, namely to get that $E(X_i X_j X_k X_\ell) = EX_i \, EX_j \, EX_k \, EX_\ell$ if i, j, k, ℓ are all distinct, and so on.

2: (25=13+12 points) Let $X_1, X_2, \ldots$ be i.i.d. **standard normal** random variables (mean 0 and variance 1). It is known by Theorem 1.1.4 on page 6 of Durrett that

$$(6x)^{-1}e^{-x^2/2} \leq P(X_i \geq x) \leq (2x)^{-1}e^{-x^2/2} \qquad \text{when } x > 2.$$

(a) Prove $\boxed{\limsup_n (X_n/\sqrt{2\log n}) \leq 1}$ a.s.

Solution. Let $\varepsilon > 0$. Then

$$\begin{aligned}\sum_{n\geq 3} P\left(\frac{X_n}{\sqrt{2\log n}} \geq \sqrt{1+\varepsilon}\right) &= \sum_{n\geq 3} P\left(X_n \geq \sqrt{2(1+\varepsilon)\log n}\right) \\ &\leq \sum_{n\geq 3} \frac{1}{2\sqrt{2(1+\varepsilon)\log n}} \exp\left(-2(1+\varepsilon)(\log n)/2\right) \\ &\leq \sum_{n\geq 3} n^{-(1+\varepsilon)} < \infty\end{aligned}$$

where we have used in the denominator that $\log n > 1$ when $n \geq 3$.

Borel–Cantelli I (Lemma 1.6.1 on page 46) now implies that

$$P\left(\frac{X_n}{\sqrt{2\log n}} \geq \sqrt{1+\varepsilon} \;\text{ i.o.}\right) = 0,$$

so that $P(\limsup_n (X_n/\sqrt{2\log n}) > \sqrt{1+\varepsilon}) = 0$. Hence $\limsup_n (X_n/\sqrt{2\log n}) \leq \sqrt{1+\varepsilon}$ a.s., and now letting $\varepsilon \downarrow 0$ through a discrete sequence of values shows that $\limsup_n (X_n/\sqrt{2\log n}) \leq 1$ a.s.

Note we have not used independence of the X_i, in Borel–Cantelli I.

(b) Prove $\boxed{\limsup_n (X_n/\sqrt{2\log n}) \geq 1}$ a.s.

Solution. We don't need to use $(1-\varepsilon)$ in this proof, because we can simply show:

$$\begin{aligned}\sum_{n\geq 2} P\left(\frac{X_n}{\sqrt{2\log n}} \geq 1\right) &= \sum_{n\geq 2} P\left(X_n \geq \sqrt{2\log n}\right) \\ &\geq \sum_{n\geq 2} \frac{1}{6\sqrt{2\log n}} \exp\left(-(2\log n)/2\right) \\ &= \frac{1}{6\sqrt{2}} \sum_{n\geq 2} \frac{1}{n\sqrt{\log n}} \\ &\geq \frac{1}{6\sqrt{2}} \int_2^\infty \frac{1}{x\sqrt{\log x}}\, dx = \infty.\end{aligned}$$

Borel–Cantelli II (Lemma 1.6.6 on page 49) now implies that

$$P\left(\frac{X_n}{\sqrt{2\log n}} \geq 1 \text{ i.o.}\right) = 1,$$

so that $\limsup_n (X_n/\sqrt{2\log n}) \geq 1$ a.s.

Note we relied on independence of the X_i when we used Borel–Cantelli II.

(c) *Extra credit, 10 points.* Prove $\boxed{\limsup_n (S_n/\sqrt{2n\log n}) \le 1}$ a.s., where $S_n = X_1 + \cdots + X_n$.

WARNING: part (c) originally had a typo, stating that "$\ldots = 1$ a.s." Here we correctly prove "$\ldots \le 1$ a.s."

Solution. S_n is a sum of independent normal random variables, and hence is itself a normal random variable, with mean $ES_n = EX_1 + \cdots + EX_n = 0$, and variance $\text{Var}(S_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n) = n$ (using independence of the X_i). Thus $S_n/\sqrt{n}$ is normal with mean 0 and variance 1.

Now part (a) above gives $\limsup_n (S_n/\sqrt{n})/\sqrt{2\log n} \le 1$ a.s. [Note we cannot use part (b) because the S_n are dependent; exercise!]

Remark. A more precise result than part (c) is the **Law of the Iterated Logarithm** (Theorem 7.9.7 on page 434):

$$\limsup_n \frac{S_n}{\sqrt{2n\log\log n}} = 1 \quad \text{a.s.}$$

So what does Problem 2 tell us?

(a) and (b) together say that if you repeatedly take independent measurements of a quantity that has a standard normal distribution, then in the long run you expect to get arbitrarily large measurements. More precisely, after about n measurements you expect to observe some measurements as large as $\sqrt{2\log n}$.

This might seem surprising, since the probability of a standard normal random variable taking a value even greater than 3 is only 0.0014. But maybe it is not so amazing after all, because when $n = 10^6$ we calculate $\sqrt{2\log n} \approx 5.7$.

As applied mathematicians like to say, the logarithm is almost a constant function.

For part (c), consider a random walk on $\mathbb{R}$ that starts at the origin, with the size of each step chosen from the standard normal distribution. Then S_n gives the position after n steps. Part (c) says that in the long run, you expect the random walk to get distance at most $\sqrt{2n\log n}$ away from the origin after n steps.

Do problem 3 or problem 4, but not both.

3: (25=15+10 points) Let $f : [0,1] \to \mathbb{R}$ be Borel measurable with $\int_0^1 f(u)^2\,du < \infty$. Let $U_1, U_2, \ldots$ be i.i.d. **uniform** random variables on $[0,1]$. Let

$$I_n = \frac{f(U_1) + \cdots + f(U_n)}{n}.$$

(a) Show $I_n \to I := \int_0^1 f(u)\,du$ a.s.

Solution. $f(U_i)$ is an L^1 random variable (in fact, L^2), with

$$\begin{aligned} E|f(U_i)| &\overset{(1.3.9)}{=} \int_0^1 |f(u)|\,du && \text{since } U_i \text{ is uniform on } [0,1] \\ &\le \int_0^1 f(u)^2\,du && \text{by Jensen's inequality} \\ &< \infty. \end{aligned}$$

Hence the expectation $Ef(U_i) = \int_0^1 f(u)\,du = I$ is well defined.

And $f(U_1), f(U_2), \ldots$ are independent, because functions of independent random variables are independent (Corollary 1.4.5), and they are also identically distributed of course. Therefore

$$I_n = \frac{f(U_1) + \cdots + f(U_n)}{n} \to Ef(U_i) = I \quad \text{a.s.}$$

by the L^1 Strong Law of Large Numbers (Theorem 1.7.1 on page 55).

(b) Let $a > 0$. Use Chebyshev's inequality to show

$$P(|I_n - I| > an^{-1/2}) \le a^{-2}\text{Var}(f(U_i)).$$

Solution.

$$\begin{aligned} P(|I_n - I| > an^{-1/2}) &\le (an^{-1/2})^{-2} E(I_n - I)^2 && \text{by Chebyshev on page 14} \\ &= a^{-2} n \text{Var}(I_n) && \text{since } EI_n = I \\ &= a^{-2} n^{-1} \text{Var}(nI_n) \\ &= a^{-2} n^{-1} \text{Var}(\sum_{m=1}^n f(U_m)) \\ &= a^{-2} n^{-1} \sum_{m=1}^n \text{Var}(f(U_m)) && \text{using independence} \\ &= a^{-2} \text{Var}(f(U_i)) && \text{using identical distribution.} \end{aligned}$$

Incidentally, observe that $\text{Var}(f(U_i))$ is finite because $f(u)$ is in L^2.

Do problem 3 or problem 4, but not both.

4: (25=10+10+5 points) Let Z_n be a **Poisson** random variable with parameter $\lambda = n$.

(a) Show $Z_n/n \to 1$ a.s.

Solution. Decompose $Z_n = X_1 + \cdots + X_n$ as a sum of Poisson i.i.d. random variables, with each X_i having parameter $\lambda = 1$. (Recall that a sum of independent Poisson random variables is again a Poisson random variable.) Then $EX_i = 1$ and so $Z_n/n \to 1$ a.s. by the Strong Law of Large Numbers. (The L^4 Strong Law and the L^1 Strong Law both apply.)

(b) Roughly sketch a normal probability density that approximates the distribution for Z_n. Indicate the approximate location and width of your sketch.

Solution. $EX_i = 1$ and $\mathrm{Var}(X_i) = 1$, and so the Central Limit Theorem 2.4.1 applied to $X_1, X_2, \ldots$ says that the distribution of $(Z_n - n)/\sqrt{n}$ is approximately standard normal, when n is large. In other words, Z_n is approximately normal with mean n and variance n, standard deviation $\sqrt{n}$. This can be sketched roughly as:

(c) Customers arrive at a store in a Poisson fashion at a rate of 100 per hour, on average. Estimate the probability that more than 120 customers arrive during the next hour.

Solution. With $n = 100$ we have

$$\begin{aligned} P(Z_{100} > 120) &= P\left(\frac{Z_{100} - 100}{\sqrt{100}} > \frac{120 - 100}{\sqrt{100}}\right) \\ &\approx P(\chi > 2) \\ &= 1 - 0.9772 = 0.0228 \end{aligned}$$

to 4 decimal places, using the Table for the standard normal distribution χ.

5: (20 points) **Do part (a) or part (b), but not both.**

(a) Let X be a uniform random variable on the interval $[-1, 1]$. Compute the characteristic function $\varphi_X(t)$. Then show there cannot exist i.i.d. random variables Y and Z with $Z - Y = X$.

Solution.

$$\varphi_X(t) = E(e^{itX}) = \int_{-1}^{1} e^{itx} \frac{1}{2} dx = \frac{\sin t}{t}.$$

Now suppose Y and Z are i.i.d. Then

$$\begin{aligned}
\varphi_{Z-Y}(t) &= E(e^{itZ} e^{-itY}) \\
&= E(e^{itZ}) E(e^{-itY}) && \text{by independence} \\
&= E(e^{itZ}) \overline{E(e^{itY})} \\
&= E(e^{itZ}) \overline{E(e^{itZ})} && \text{by identical distribution} \\
&= |\varphi_Z(t)|^2 \geq 0.
\end{aligned}$$

This nonnegative function cannot equal $\varphi_X(t) = (\sin t)/t$, because $\sin t$ is negative sometimes (for example at $t = 3\pi/2$).

(b) Prove that if $X_n \Longrightarrow X$ and $c_n \to 0$ then $X_n - c_n \Longrightarrow X$. (Use only the definition of weak convergence; do not use any theorems or exercises.)

Solution. Let $F(x) = P(X \leq x)$ be the distribution function of X. Suppose x is a point of continuity of F. Choose $\varepsilon > 0$ such that $x + \varepsilon$ is also a point of continuity; note this works for all except countably many ε-values, because F has at most countably many discontinuities. Then

$$\begin{aligned}
P(X_n - c_n \leq x) &= P(X_n \leq x + c_n) \\
&\leq P(X_n \leq x + \varepsilon) && \text{for all large } n\text{, since } c_n \to 0 \\
&\to P(X \leq x + \varepsilon) && \text{because } X_n \Longrightarrow X.
\end{aligned}$$

That is, we have shown

$$\limsup_n P(X_n - c_n \leq x) \leq P(X \leq x + \varepsilon) = F(x + \varepsilon),$$

for all except countably many ε-values. Letting $\varepsilon \to 0$ implies

$$\limsup_n P(X_n - c_n \leq x) \leq P(X \leq x) = F(x),$$

using here that F is continuous at x.

A similar argument gives that $P(X \leq x)$ is a lower bound on the $\liminf$, and so in fact equality holds and $\lim_n P(X_n - c_n \leq x) = P(X \leq x)$, which proves that $X_n - c_n \Longrightarrow X$.

Remark. We have already used the result in part (b) several times in class; now you have a proof.