

Kaczmarz Algorithm and Frames

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Kaczmarz Algorithm

In 1937 Stefan Kaczmarz introduced the following algorithm to solve linear system $Ax = b$:

Let a_n denote n^{th} row of A , and let $x_0 \in \mathbb{C}^d$. Define

$$x_{k+1} = x_k + \frac{b_i - \langle x_k, a_i \rangle}{\|a_i\|^2} a_i,$$

where $k + 1 \equiv i \pmod{n}$. If A is of full rank, then:

$$\lim_{k \rightarrow \infty} \|x_k - x\| = 0.$$

Kaczmarz Algorithm in Infinite Dimensions

Let \mathcal{H} be a Hilbert space and let $\{e_n\}_{n=0}^{\infty}$ be a sequence of unit vectors in \mathcal{H} . Given $x \in \mathcal{H}$, the Kaczmarz algorithm is defined as:

$$x_0 = \langle x, e_0 \rangle e_0,$$

and

$$\forall n \in \mathbb{N}, \quad x_n = x_{n-1} + \langle x - x_{n-1}, e_n \rangle e_n.$$

We say the sequence $\{e_n\}_{n=0}^{\infty}$ is *effective* if and only if

$$\forall x \in \mathbb{H}, \quad \lim_{n \rightarrow \infty} x_n = x.$$

Kaczmarz Algorithm in Infinite Dimensions

In 2001 Kwapien and Mycielski introduced the sequence $\{g_n\}_{n=0}^{\infty}$:

$$g_0 = e_0,$$

and

$$\forall n \in \mathbb{N}, \quad g_n = e_n - \sum_{i=0}^{n-1} \langle e_n, e_i \rangle g_i. \quad (1)$$

Thus,

$$x_n = \sum_{i=0}^n \langle x, g_i \rangle e_i.$$

Theorem 1 (Kwapien, Mycielski (2001)). *The sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if $\{g_n\}_{n=0}^{\infty}$ is a tight frame with constant 1 for \mathcal{H} .*

Proof

We have

$$x - x_{n-1} = x - x_n + \langle x, g_n \rangle e_n.$$

Since $x - x_n$ and e_n are orthogonal,

$$\|x - x_{n-1}\|^2 = \|x - x_n\|^2 + |\langle x, g_n \rangle|^2.$$

Also,

$$\|x\|^2 = \|x - x_0\|^2 + |\langle x, g_0 \rangle|^2.$$

Summing up the above equations, we get

$$\|x\|^2 = \lim_{n \rightarrow \infty} \|x - x_n\|^2 + \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2.$$

□

Relationship between $\{e_n\}$ and $\{g_n\}$

We have

$$e_n = \sum_{i=0}^n \langle e_n, e_i \rangle g_i.$$

Equivalently, we can write

$$g_n = e_n + \sum_{i=0}^{n-1} c_{ni} e_i,$$

for some coefficients c_{ni} .

Question: How to compute c_{ni} 's?

Characterization of Effective Sequences

Let M be a lower triangular matrix

$$M(i, j) = m_{ij} = \begin{cases} \langle e_i, e_j \rangle & i > j, \\ 0 & i \leq j. \end{cases}$$

Then $(I + M)^{-1} = I + C$, where

$$C(i, j) = \begin{cases} c_{ij} & i > j, \\ 0 & i \leq j, \end{cases}$$

with c_{ij} as defined before.

Theorem 2 (Haller, Szwarc (2005)). *The sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if it is linearly dense and C is a partial isometry, i.e., C^*C is an orthogonal projection.*

Effective Sequences and Bases

Example 1. *If $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for \mathcal{H} , then $g_n = e_n$. Hence, by Theorem 1, $\{e_n\}_{n=0}^{\infty}$ is an effective sequence.*

Theorem 3. *An effective sequence is a Riesz basis if and only if it is an orthonormal basis.*

Sketch of Proof: If G denotes the Gram matrix of $\{e_n\}_{n=0}^{\infty}$, then $(I + C)^*G(I + C) = I - C^*C$. Moreover, if $\{e_n\}_{n=0}^{\infty}$ is not orthonormal then $C \neq 0$. As such C^*C is a non-zero projection and $I - C^*C$ has non-trivial kernel. Hence, there exists $x \neq 0$ such that $\langle Gx, x \rangle = 0$.

□

Bessel sequences

Haller and Szwarc showed that there exist effective sequences which are not Bessel systems. In fact, these sequences satisfy:

$$\forall j \in \mathbb{N}, \quad \sum_{i=0}^{\infty} |\langle e_i, e_j \rangle|^2 = \infty,$$

and

$\{e_n\}_{n=k}^{\infty}$ is effective for each k .

This can be compared with the fact that for every Bessel sequence $\{g_n\}_{n=0}^{\infty}$ with $\|g_0\| = 1$, there exists a sequence $\{e_n\}_{n=0}^{\infty}$ of unit vectors such that (1) holds.

Frames and Riesz Bases

Theorem 4. *Let $\{e_n\}_{n=0}^{\infty}$ be a Bessel sequence and let M be bounded. Then, $\{e_n\}_{n=0}^{\infty}$ is a frame if and only if $\{g_n\}_{n=0}^{\infty}$ is a frame.*

Theorem 5. *Let $\{e_n\}_{n=0}^{\infty}$ be a Bessel sequence and M be bounded. Then $\{e_n\}_{n=0}^{\infty}$ is a Riesz basis if and only if $\{g_n\}_{n=0}^{\infty}$ is a Riesz basis.*

Orthonormal Bases

Corollary 1. *Let $\{e_n\}_{n=0}^{\infty}$ be a Bessel sequence and M be bounded. Then $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis if and only if $\{g_n\}_{n=0}^{\infty}$ is an orthonormal basis.*

Proof: Clearly, if $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis, then so is $\{g_n\}_{n=0}^{\infty}$. For the other direction, if $\{g_n\}_{n=0}^{\infty}$ is an orthonormal basis, then it is a Riesz basis and a 1-tight frame. Thus, by Theorem 1 and Theorem 5, $\{e_n\}_{n=0}^{\infty}$ is a Riesz basis that is also effective. Theorem 3 implies that it is an orthonormal basis.

□

Almost Effective Sequences

Recall that an effective sequence $\{e_n\}_{n=0}^{\infty}$ yields

$$\lim_{n \rightarrow \infty} \|x - x_n\| = 0.$$

A sequence $\{e_n\}_{n=0}^{\infty}$ is said to be *almost effective*, with constants $0 \leq A \leq B < 1$, if

$$\forall x \in \mathcal{H}, \quad A\|x\|^2 \leq \lim_{n \rightarrow \infty} \|x - x_n\|^2 \leq B\|x\|^2.$$

Example 2. *Following the result of Kwapien and Mycielski, sequence $\{e_n\}$ is almost effective with bounds $0 \leq A \leq B < 1$ if and only if $\{g_n\}$ is a frame with bounds $0 < (1 - B) \leq (1 - A) \leq 1$.*

Characterization of Almost Effective Sequences

Proposition 1. *Let $\{e_n\}_{n=0}^{\infty}$ be a Bessel sequence. Then, if $\{e_n\}_{n=0}^{\infty}$ is almost effective, it is also a frame.*

If, in addition, M is bounded then the converse holds as well.

Given the sequence $\{e_n\}_{n=0}^{\infty}$, let $L(x)(n) = \langle x, e_n \rangle$.

Theorem 6. *Let $\{e_n\}_{n=0}^{\infty}$ be a Bessel sequence. Then $\{e_n\}_{n=0}^{\infty}$ is almost effective if and only if it is a frame and $\overline{\text{ran}((I + C)|_{\text{ran}(L)})}$ is closed.*

Summary of $\{e_n\} \leftrightarrow \{g_n\}$ Correspondence

$\{e_n\}$ effective $\leftrightarrow \{g_n\}$ 1-tight frame

$\{e_n\}$ almost effective $\leftrightarrow \{g_n\}$ frame

(With additional assumptions:)

$\{e_n\}$ frame $\leftrightarrow \{g_n\}$ frame

$\{e_n\}$ Riesz basis $\leftrightarrow \{g_n\}$ Riesz basis

$\{e_n\}$ ONB $\leftrightarrow \{g_n\}$ ONB

Summary of Almost Effective Sequences

If $\{g_n\}$ is a frame, then $\{e_n\}$ is almost effective.

If $\{e_n\}$ is a frame and M is bounded then $\{e_n\}$ is almost effective.

If $\{e_n\}$ is a frame and $\overline{\text{ran}((I + C)|_{\text{ran}(L)})}$ is closed, then $\{e_n\}$ is almost effective.