

Maximal multipliers in L^p and improved range
in the Return Times theorem

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Consider a finite set $\Lambda = \{\lambda_1, \dots, \lambda_N\} \subset \mathbb{R}$. For each $k \in \mathbb{Z}$ define R_k to be the collection of all intervals of length 2^k centered at some element from Λ . The following result is due to Bourgain, 1988.

Lemma (Bourgain's Lemma). *For each $f \in L^2(\mathbb{R})$,*

$$\left\| \sup_k \left| \int_{R_k} \widehat{f}(\xi) e^{2\pi i \xi x} d\xi \right| \right\|_{L^2_x} \lesssim (\log N)^2 \|f\|_2$$

The case $N = 1$ is a variant of the Hardy-Littlewood maximal function.

Using Kolmogorov's rearrangement theorem for series, Bourgain, Kostyukovsky and Olevskii, 2001, proved that at least $(\log N)^{1/4}$ is needed. The exact growth of the best constant is not known.

Let $\mathbf{X} = (X, \Sigma, \mu, T)$ be a dynamical system, that is a probability space (X, Σ, μ) equipped with an invertible bimeasurable measure preserving transformation $T : X \rightarrow X$ ($A \in \Sigma \Rightarrow \mu(A) = \mu(T^{-1}A)$).

Theorem 1 (Bourgain, 1988). *Let $f \in L^p(X)$, $p > 1$. Then the averages*

$$\frac{1}{N} \sum_{n=1}^N f(T^{n^2} x)$$

converge almost everywhere.

Theorem 2 (Bourgain, 1990). *Let $f, g \in L^\infty(X)$. Then the bilinear averages*

$$\frac{1}{N} \sum_{n=1}^N f(T^n x) g(T^{-n} x)$$

converge almost everywhere.

Theorem 3 (Lacey, 2000). *Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, with $1 < p, q \leq \infty$ and $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} < \frac{3}{2}$. Then the bilinear maximal function*

$$BM(f, g)(x) := \sup_{\epsilon > 0} \left| \frac{1}{\epsilon} \int_{|t| < \epsilon} f(x+t) g(x-t) dt \right|$$

maps $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$

The following Return Times Theorem was proved by D., Lacey, Thiele and Tao in 2008

Theorem 4. *Let $\mathbf{X} = (X, \Sigma, \mu, \tau)$ be a dynamical system, and let $1 < q \leq \infty$ and $p \geq 2$. For each function $g \in L^q(X)$ there is a universal set $X_0 \subseteq X$ with $\mu(X_0) = 1$, such that for each second dynamical system $\mathbf{Y} = (Y, \mathbb{F}, \nu, \sigma)$, each $f \in L^p(Y)$ and each $x \in X_0$, the averages*

$$\frac{1}{N} \sum_{n=1}^N g(\tau^n x) f(\sigma^n y)$$

converge for ν -almost every y .

The way we proved this result was by relating it to Carleson's theorem on pointwise convergence of Fourier series. The methods are also somewhat related to the argument of Lacey and Thiele [1997,1999] for the boundedness of the Bilinear Hilbert Transform

$$BHT(f_1, f_2)(x) = p.v. \int_{\mathbb{R}} f_1(x+t) f_2(x-t) \frac{dt}{t}.$$

Another key ingredient was a weighted version of Bourgain's Lemma

For each $1 \leq r < \infty$ and each sequence $(x_k)_{k \in \mathbb{Z}} \in \mathbb{C}$, define the r -variational norm of $(x_k)_{k \in \mathbb{Z}}$ to be

$$\|x_k\|_{V_k^r} := \sup_k |x_k| + \|x_k\|_{\tilde{V}_k^r}$$

where \tilde{V}_k^r is the homogeneous r -variational seminorm

$$\|x_k\|_{\tilde{V}_k^r} := \sup_{M, k_0 < k_1 < \dots < k_M} \left(\sum_{m=1}^M |x_{k_m} - x_{k_{m-1}}|^r \right)^{1/r}.$$

For each interval $\omega \in R_k$, let $(m_\omega)_{\omega \in R_k} : \omega \rightarrow \mathbb{C}$ be a multiplier supported in and adapted to ω , that is

$$\|m_\omega^{(\alpha)}\|_\infty \leq |\omega|^{-\alpha}, \quad \alpha \in \{0, 1\}.$$

Define

$$\Delta_k f(x) := \sum_{\omega \in R_k} \int m_\omega(\xi) \widehat{f}(\xi) e^{2\pi i \xi x} d\xi,$$

and also, with the notation $\mathcal{M} := \{m_\omega\}$

$$\|\mathcal{M}\|_{V^{r,*}} := \max_{1 \leq n \leq N} \|\{m_{\omega_k}(\lambda_n) : \lambda_n \in \omega_k \in R_k\}\|_{V_k^r},$$

Lemma. *For each $r > 2$ and $f \in L^2(\mathbb{R})$ we have the inequality*

$$\|\sup_k |\Delta_k f(x)|\|_{L_x^2(\mathbb{R})} \lesssim_r N^{1/2-1/r} (1 + \|\mathcal{M}\|_{V^{r,*}}) \|f\|_2,$$

with the implicit constant depending only on r .

This is applied with r sufficiently close to 2. It is an L^2 result, and it restricts applicability in the Return Times theorem to the case when the test function f is in $L^p(X)$, with $p \geq 2$.

Question. *What are the best constants in the above lemma, when $p \neq 2$?*

Theorem 5 (D., preprint 2008). *For each $1 < p < \infty$, each $\epsilon > 0$ and each $r > 2$ we have the inequality*

$$\| \sup_k |\Delta_k f(x)| := \sum_{\omega \in R_k} \int m_\omega(\xi) \widehat{f}(\xi) e^{2\pi i \xi x} d\xi \|_{L_x^p(\mathbb{R})} \lesssim N^{|\frac{1}{p} - \frac{1}{r} + \epsilon|} (1 + \|\mathcal{M}\|_{V^{r,*}}) \|f\|_p$$

with the implicit constant depending only on r , ϵ and p .

The exponent of N is *essentially sharp*, as r approaches 2.

Proposition 6. *For each $N \in \mathbb{N}$ and $p \in (1, 2)$ there is a choice of signs $(\epsilon_n)_{1 \leq n \leq N}$ such that if $\widehat{f}_N = 1_{[0, N]}$ then*

$$\| \int \widehat{f}_N(\xi) \sum_{l=0}^{N-1} \epsilon_n 1_{[n, n+1]}(\xi) e^{2\pi i \xi x} d\xi \|_{L_x^p(\mathbb{R})} \gtrsim N^{\frac{1}{p} - \frac{1}{2}} \|f_N\|_{L^p(\mathbb{R})}.$$

Proof. It immediately follows that

$$\|f_N\|_{L^p(\mathbb{R})} \sim N^{1-1/p},$$

$$\| \left(\sum_{n=0}^{N-1} \left| \int \widehat{f}_N(\xi) 1_{[n, n+1]}(\xi) e^{2\pi i \xi x} d\xi \right|^2 \right)^{1/2} \|_{L_x^p(\mathbb{R})} \sim N^{1/2}.$$

Khintchine's inequality ends the proof. □

Using this result, one can improve the range in the Return Times Theorem

Theorem 7 (D., 2009). Let $\mathbf{X} = (X, \Sigma, \mu, \tau)$ be a dynamical system, and let $1 \leq p, q \leq \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} < \frac{3}{2}.$$

For each function $g \in L^q(X)$ there is a universal set $X_0 \subseteq X$ with $\mu(X_0) = 1$, such that for each second dynamical system $\mathbf{Y} = (Y, \mathbb{F}, \nu, \sigma)$, each $f \in L^p(Y)$ and each $x \in X_0$, the averages

$$\frac{1}{N} \sum_{n=1}^N g(\tau^n x) f(\sigma^n y)$$

converge for ν -almost every y .

Interestingly, this is exactly the range where the bilinear Hilbert transform, and the bilinear maximal function [Lacey 2000]

$$BM(f_1, f_2)(x) = \sup_{t>0} \frac{1}{2t} \left| \int_{-t}^t f_1(x+y) f_2(x-y) dy \right|$$

are known to be bounded. In both cases, the methods fail beyond the $\frac{3}{2}$ threshold essentially because of the same reason.

Both operators are known to be unbounded for pairs of L^1 functions, in quite a dramatic way: even their tails are unbounded!!

$$\sup_{t>1} \frac{1}{2t} \left| \int_t^{t+1} f_1(x+y) f_2(x-y) dy \right|$$

Theorem 8 (D., preprint 2008). *For each $1 < p < \infty$, each $\epsilon > 0$ and each $r > 2$ we have the inequality*

$$\| \sup_k |\Delta_k f(x)| := \sum_{\omega \in R_k} \int m_\omega(\xi) \widehat{f}(\xi) e^{2\pi i \xi x} d\xi \|_{L_x^p(\mathbb{R})} \lesssim N^{|\frac{1}{p} - \frac{1}{r} + \epsilon|} (1 + \|\mathcal{M}\|_{V^{r,*}}) \|f\|_p$$

with the implicit constant depending only on r , ϵ and p .

Proof. (of the case easy case $p > 2$). Assume λ_n are 1 separated and $2^k \ll 1$. By interpolating with the L^2 result, it suffices to achieve the crude bound $N^{1/2}$ for each $p > 2$. By Cauchy-Schwartz

$$\sup_{2^k \ll 1} |\Delta_k f(x)| \leq N^{1/2} \left(\sum_{n=1}^N (M(f_n)(x))^2 \right)^{1/2}, \text{ where}$$

$$f_n(x) = \int_{|\xi - \lambda_n| < 1} \widehat{f}(\xi) e^{2\pi i \xi x} d\xi,$$

and $M(f)$ denotes the Hardy-Littlewood maximal function of f . The result then follows from a combination of the Fefferman-Stein inequality and Rubio de Francia's result on the L^p boundedness ($p > 2$) of the square function

$$SQ(f)(x) := \left(\sum_{n=1}^N |f_n(x)|^2 \right)^{1/2}.$$

□

The proof in the case $p < 2$ involves the following main steps:

1. Estimates for exponential sums:

Lemma (Bourgain). *For each set $C = \{c_k\} \subseteq l^2([N])$ and each $r > 2$ we have*

$$\left\| \sup_k \left| \sum_{n=1}^N c_{k,n} e^{2\pi i \lambda_n y} \right| \right\|_{L_y^2([0, D^{-1}])} \lesssim N^{1/2-1/r} \|c_k\|_{V_k^r(N)},$$

with the implicit constant depending only on r .

2. Define for each dyadic interval I , $h_{I,n}(x) = \frac{1}{|I|} \mathbf{1}_{[0,1]} \left(\frac{x-c(I)}{|I|} \right) e^{2\pi i \lambda_n x}$.

For $r > 2$ define

$$Vf(x) := \left(\sum_{n=1}^N \|\mathbf{1}_I(x) \langle f, h_{I,n} \rangle\|_{V_I^r}^2 \right)^{1/2}.$$

Theorem 9. *We have for each $1 < p \leq 2$ and each $\epsilon > 0$*

$$\|Vf\|_p \lesssim N^{\frac{1}{p}-\frac{1}{2}+\epsilon} \|f\|_p, \tag{1}$$

with the implicit constant depending only on r , p and ϵ .

The proof requires Time-frequency analysis and vector valued martingale theory.