On curvature and the bilinear multiplier problem

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Fefferman’s Ball Multiplier Theorem

\[ B_d = \text{unit ball of } \mathbb{R}^d \]

\textbf{Theorem (C. Fefferman, 1971)}

For \( d \geq 2 \), the Fourier multiplier operator with symbol \( \chi_{B_d} \) is bounded on \( L^p(\mathbb{R}^d) \) only when \( p = 2 \).

\textbf{Contrast with:}

- \( d = 1 \): \( \chi_{B_1} \) is a bounded Fourier multiplier on \( L^p(\mathbb{R}) \) for all \( 1 < p < \infty \).
- \( d \geq 2 \): If \( D \) is a “polyhedral” domain (with finitely many faces), then \( \chi_D \) is a bounded Fourier multiplier on \( L^p(\mathbb{R}^d) \) for all \( 1 < p < \infty \).

\textbf{Moral:} No curvature in either of these cases.
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Moral: No curvature in either of these cases.
Curvature

From now on, $D \subset \mathbb{R}^d$ denotes a domain with $\partial D$ a piecewise-smooth hypersurface (codimension-1).

**Dichotomy**

- If $\partial D$ has some nonzero sectional curvature at some point, then $\chi_D$ is an $L^p$-bounded Fourier multiplier only in the trivial case $p = 2$.
- Otherwise, $\partial D$ is flat, and $D$ is a polyhedral domain as above.

**Question**

Does such a clear dichotomy persist in an appropriate bilinear setting?

**Answer**

Not as precisely.

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Bilinear Fourier multipliers

$D \subset \mathbb{R}^{2d}$ a domain. Bilinear Fourier multiplier on $S(\mathbb{R}^d) \times S(\mathbb{R}^d)$:

$$T_D(f, g)(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_D(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i (\xi+\eta) \cdot x} \, d\xi \, d\eta \in S(\mathbb{R}^d).$$

Associated trilinear form on $S(\mathbb{R}^d) \times S(\mathbb{R}^d) \times S(\mathbb{R}^d)$:

$$\Lambda_D(f_1, f_2, f_3) = \int_{\mathbb{R}^d} T_D(f_1, f_2)f_3$$

So

$$\| T_D(f, g) \|_{p_3'} \lesssim \| f \|_{p_1} \| g \|_{p_2} \iff |\Lambda_D(f_1, f_2, f_3)| \lesssim \prod_{j=1}^{3} \| f_j \|_{p_j}.$$
What can we say about boundedness of $\Lambda_{B_{2d}}$ on $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \times L^{p_3}(\mathbb{R}^d)$ with $\sum \frac{1}{p_j} = 1$?

### $d=1$

- **Local-$L^2$** ($p_1, p_2, p_3 \geq 2$): $\Lambda_{B_2}$ is bounded (Grafakos–Li, 2006).
- **Outside local-$L^2$** ($p_j < 2$ for exactly one $j$): ?

### $d \geq 2$

- **Local-$L^2$**: ?
- **Outside local-$L^2$**: $\Lambda_{B_{2d}}$ is unbounded (Diestel–Grafakos, 2007). How generally does the presence of curvature yield unboundedness?
What can we say about boundedness of $\Lambda_{B_{2d}}$ on $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \times L^{p_3}(\mathbb{R}^d)$ with $\sum \frac{1}{p_j} = 1$?

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For now, restrict attention to $d = 2$ (so that we deal with domains in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$).

**Theorem (Grafakos–Reguera, 2008)**

*If $D \subset \mathbb{R}^4$ is a compact, strictly convex domain, then the trilinear form $\Lambda_D$ is unbounded outside the local-$L^2$ setting.*
Why the global hypothesis?

- Proof requires $\partial D$ to have normal vectors of the form $(\nu, \nu)$ for a rich class of $\nu \in S^1$. Use surjectivity of the Gauss map on $\partial D$ to guarantee these exist.
- Method doesn’t really need $(\nu, \nu)$, only $(\nu, \lambda \nu)$.
- “Diagonal Gauss map condition”: Range of the Gauss map on $\partial D$ must have “large” intersection with the diagonal of $\mathbb{R}P^1 \times \mathbb{R}P^1$ when projectivized in each coordinate.
- NB: This condition is not $SO(\mathbb{R}^4)$-invariant and is not guaranteed by local convexity.

This method cannot treat the operator with symbol given by $B_2 \times \mathbb{R}^2$, which is clearly unbounded.
Question
Is nontrivial sectional curvature at a boundary point enough to guarantee unboundedness?

Answer
No. There are domains $D_0 \subset \mathbb{R}^1 \times \mathbb{R}^1$ with curved boundary for which $\Lambda_{D_0}$ is bounded outside local $L^2$. (Muscalu, 2000)

Embed such a $D_0$ as a suitably-placed cylinder set in $\mathbb{R}^2 \times \mathbb{R}^2$:

$$D = \{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \mid (\xi_1, \xi_3) \in D_0\}.$$
An alternate approach

Notational convenience:

Recast $D \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ inside $\mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$:

- $\Gamma := \{\xi_1 + \xi_2 + \xi_3 = 0\}$ is an isomorphic copy of $\mathbb{R}^2 \times \mathbb{R}^2$ embedded in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ via the map

$$
(\xi_1, \xi_2) \mapsto (\xi_1, \xi_2, -(\xi_1 + \xi_2)).
$$

- Let $\tilde{D}$ denote the image of $D$ under this embedding. Then

$$\Lambda_D(f_1, f_2, f_3) = \int_{\Gamma} \chi_{\tilde{D}} \hat{f}_1 \otimes \hat{f}_2 \otimes \hat{f}_3 \, d\sigma,$$

where $\sigma$ is the surface measure on the subspace $\Gamma$.

- NB: The intersection of $\Gamma$ with any section of the form $\{(\xi_1, \xi_2, \xi_3) \mid \xi_j = \xi_0\}$ is a 2-plane.
An alternate approach: sectional curvature conditions

Main Theorem

Let $D \subset \mathbb{R}^4$ be a domain and $\tilde{D} \subset \Gamma$ as before. Suppose that at some point $\partial \tilde{D}$ has nonzero sectional curvature in a plane of the form

$$\{(\xi_1, \xi_2, \xi_3) \in \Gamma \mid \xi_j = \xi_0\}.$$  

Then the trilinear form $\Lambda_D$ is unbounded on $L^{p_1} \times L^{p_2} \times L^{p_3}(\mathbb{R}^2)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ with $p_i < 2$ for some $i \neq j$.

- The hypothesis here is again “rotation-rigid.”
- If the $j$-th sectional curvature condition above is satisfied for two choices of $j \in \{1, 2, 3\}$, we obtain full unboundedness outside local-$L^2$. 

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- If the $j$-th sectional curvature condition above is satisfied for two choices of $j \in \{1, 2, 3\}$, we obtain full unboundedness outside local-$L^2$. 
Corollary

If \( D \subset \mathbb{R}^4 \) is \textit{locally} strictly convex near some point of \( \partial D \), then \( \Lambda_D \) is unbounded outside local-L\(^2\).
Recall Fefferman’s proof

- Assume toward a contradiction that $T_{B_2} : L^p \to L^p$ for some $1 < p < 2$.
- The boundary of the disc $B_2$ has tangent lines in all directions, so we can approximate any half-plane by suitable dilates and translates of $B_2$:

  ![Approximation Diagram]

- Use this to get square function estimates for half-plane multipliers:
Recall Fefferman’s proof

- Square function estimate: For any sequence of directions $\nu_j \in S^1$,

$$\left\| \left( \sum_j |T_{\nu_j} f_j|^2 \right)^{1/2} \right\|_p \lesssim_p \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_p.$$

$T_{\nu_j}$ is the half-plane multiplier with symbol the characteristic function of $\{\xi \in \mathbb{R}^2 \mid \xi \cdot \nu_j > 0\}$.

- Take a Kakeya set consisting of thin rectangles $R_j$ of length 1 pointing in the directions $\nu_j$. Then

$$|T_{\nu_j} \chi_{R_j}| \gtrsim \chi_{R'_j},$$

where the rectangles $R'_j$ are pairwise disjoint.
Recall Fefferman’s proof

- All that’s important about the disc here is that $\partial B_2$ has enough normal vectors to yield Kakeya sets.

- What happens when we apply Fefferman’s method in the bilinear setting?
  - Assume boundedness of $T_D$ and obtain square function estimates for half-space bilinear multipliers.
  - Try to exploit Kakeya sets.
Action of half-space bilinear multipliers on rectangles

- \( \vec{v} = (v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \) direction vector, \( P_{\vec{v}} = \{ \vec{\xi} \cdot \vec{v} > 0 \} \) half-space in \( \mathbb{R}^2 \times \mathbb{R}^2 \).

- \( T_{P_{\vec{v}}}(f, g)(x) \approx \int_{\mathbb{R}} f(x - tv_1) g(x - tv_2) \frac{dt}{t} \).

- \( R_j \) rectangles oriented parallel to \( v_j \). \( T_{P_{\vec{v}}}(\chi_{R_1}, \chi_{R_2}) \) is supported on an intersection of strips:
“Reach” of a rectangle $R$:

\begin{align*}
R & \quad \quad \quad \quad \quad \quad R' \\
\text{\includegraphics[width=0.5\textwidth]{rectangle.png}}
\end{align*}

We want to recover the reach of $R$ via $T_{P_{\mathcal{V}}}$:

$$
|T_{P_{\mathcal{V}}} (\chi_R, \chi_S)| \gtrsim \chi_{R'}
$$

for some other rectangle $S$. 
One way:

If we want $|T_{P_{\frac{v}{v}}} (\chi_R, \chi_S)| \gtrsim \chi_R'$ and insist that $S$ have the same dimensions as $R$, then the strips associated to $R$ and $S$ must actually coincide:

So we must have $v_1$ parallel to $v_2$; i.e.

$$\vec{v} = (v, \lambda v).$$
Another way:

On the other hand, if $S$ is allowed to be large relative to $R$, we can achieve $|T_{P_{\vec{v}}}(\chi_R, \chi_S)| \gtrsim \chi_{R'}$ with no restriction on $\vec{v}$:

Heuristic limiting case: $S = \mathbb{R}^2$, $T_{P_{\vec{v}}}(f, \chi_S)$ is just the directional Hilbert transform with direction $\vec{v}_1$. 

\[
\begin{array}{c}
R \\
\vdots \\
S \\
\vdots \\
R' \\
\vdots \\
\end{array}
\]

\[
\begin{array}{c}
v_1 \\
\vdots \\
v_2 \\
\vdots \\
\end{array}
\]
Square function estimates dictate geometry

Assume $T_D : L^{p_1} \times L^{p_2} \to L^{p_3'}$ with $p_3 < 2$. $\vec{v}_j = (v_j^1, v_j^2)$ normal vectors to $\partial D$.

Take disjoint rectangles $R_j$ pointing in the direction of $v_1^j$, such that $\sum |R_j| = 1$ while $|\bigcup R_j'|$ is arbitrarily small.

If we have $|T_{\vec{v}_j} (\chi_{R_j}, \chi_{S_j})| \gtrsim \chi_{R_j'}$, we can exploit two distinct square function estimates:

1. $\left\| \left( \sum_j |T_{\vec{v}_j} (\chi_{R_j}, \chi_{S_j})|^2 \right)^{1/2} \right\|_{p_3'} \lesssim \left\| \left( \sum_j |\chi_{R_j}|^2 \right)^{1/2} \right\|_{p_1} \left\| \left( \sum_j |\chi_{S_j}|^2 \right)^{1/2} \right\|_{p_2}$

   • Here we need $\sum |S_j| \sim 1$, so we need $\vec{v}_j = (v_j, \lambda_j v_j)$ (diagonal Gauss map condition).

2. $\left\| \left( \sum_j |T_{\vec{v}_j} (\chi_{R_j}, \chi_{S_j})|^2 \right)^{1/2} \right\|_{p_3'} \lesssim \| \chi_S \|_{p_2} \left\| \left( \sum_j |\chi_{R_j}|^2 \right)^{1/2} \right\|_{p_1}$

   • No restriction on $\vec{v}_j$ here, but the points at which $\vec{v}_j$ occur must lie in a coordinate slice (sectional curvature condition).
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   No restriction on $\vec{v}_j$ here, but the points at which $\vec{v}_j$ occur must lie in a coordinate slice (sectional curvature condition).
Multiplier theorems for balls
More general domains

Square function estimates dictate geometry

Assume $T_D : L^{p_1} \times L^{p_2} \rightarrow L^{p_3'}$ with $p_3 < 2$. $\overrightarrow{v}_j = (v^1_j, v^2_j)$ normal vectors to $\partial D$.

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If we have $|T_P \overrightarrow{v}_j (\chi_{R_j}, \chi_{S_j})| \gtrsim \chi_{R'_j}$, we can exploit two distinct square function estimates:

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Symmetry

\[ M_{p,q} \]: Class of \textit{linear} \( L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) Fourier multipliers

\[ M^b_{p,q,r} \]: Class of \textit{bilinear} \( L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d) \) Fourier multipliers

Consider symmetries of these classes under actions on the underlying Euclidean space:

- \( M_{p,q} \) is invariant under the \textit{full} affine group \( \text{GL}_d(\mathbb{R}) \rtimes \mathbb{R}^d \).
- \( M^b_{p,q,r} \) is not even invariant under \( \text{SO}_{2d}(\mathbb{R}) \):

Take a domain \( D \subset \mathbb{R}^4 \) such that \( T_D \) is bounded and \( \partial D \) has nontrivial sectional curvature. Rotate it so that it now has curvature in an appropriate coordinate slice.
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Take a domain \( D \subset \mathbb{R}^4 \) such that \( T_D \) is bounded and \( \partial D \) has nontrivial sectional curvature. Rotate it so that it now has curvature in an appropriate coordinate slice.
Open problems

- Produce a domain $D$ whose boundary satisfies the diagonal Gauss map condition but fails to satisfy our sectional curvature conditions.

- Examples of $D$ where $\partial D$ fails both conditions (diagonal Gauss map and all $j$-th sectional curvature conditions):

$$\{ \vec{\xi} \in \mathbb{R}^4 \mid \xi_4 > \xi_1\xi_3 + \xi_1^2 \}.$$  

What happens for these?

- What happens inside local-$L^2$?

- Say anything about the half-space operators $T_{P_{\vec{v}}}$ (two-dimensional bilinear Hilbert transforms) inside the Banach triangle.
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- What happens inside local-$L^2$?
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