

Spline Wavelets, Finite Element Wavelets, and Wavelets with Composite Dilation

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Construction of Multivariate Wavelets on Arbitrary Triangulation

- ▶ Haar-type non-separable constant wavelets: “twin dragon,” Belogay and Wang [99], Flaherty and Wang [99], and Gröchenig and Madych [92]; wavelet with composite dilations, Krishtal, Robinson, Weiss, and Wilson [08]
- ▶ Continuous piecewise linear wavelet: Yserençant [86], Vassilevski and Wang [97], Stevenson [97,98, H^1 -stable], Liu [06], Floater and Quak [99,00, semi-orthogonal], Hardin and Hong [03, orthogonal on type-1 triangulation]
- ▶ C^1 quadratic splines and spline wavelets: Powell-Sabin [77], Chui and He[90], Chui, Chui, and He[93], Chui and Jiang [04]; Oswald [92, not H^2 -stable], Davydov and Petrushev [03,05, H^μ -stable, $\mu \in (1, 5/2)$] ($\|f\|_{H^\mu(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^{2\mu} d\xi \right)^{1/2} < \infty$), Windmolders, Vanraes, Dierckx, and Bultheel [03].

Splines and Elements, Spline Wavelets, Wavelets with Composite Dilations

- ▶ Splines and their BB-expressions: Farin [88,90,93], Chui [87], etc.
- ▶ Characterization of compactly supported refinable splines: Lawton, Lee, and Shen [95], Sun [96], Goodman [98], Guan and He [09], etc.
- ▶ Spline wavelets: Chui and Wang [92,93], Chui, Stökler, and Ward [92], Jia and Micchelli [91], Riemenschneider and Shen [92], Lorentz and Oswald [00, Sobolev spaces], Jia, Wang, and Zhou [03], Jia and Liu [08], etc.
- ▶ Wavelets with composite dilations: Guo, Labate, Lim, Weiss, and Wilson [04, 06, 06], etc.

BB-expressions of polynomials and splines-1

Let $x^0, \dots, x^d \in \mathbb{R}^d$, $d \geq 1$, $x^i = (x_1^i, \dots, x_d^i)$ and consider the convex hull

$$T_d := \langle x^0, \dots, x^d \rangle = \left\{ \sum_{i=0}^d \alpha_i x^i : \sum_{i=0}^d \alpha_i = 1, \alpha_i \geq 0 \right\}.$$

This convex hull is called an d -simplex if its signed volume $\text{Vol}_d \langle x^0, \dots, x^d \rangle$ is nonzero. Suppose that $\langle x^0, \dots, x^d \rangle$ is an d -simplex. Then any $x \in \mathbb{R}^d$ can be identified by an $(d+1)$ -tuple $\lambda = (\lambda_0, \dots, \lambda_d)$, the *barycentric coordinates* of x relative to the d -simplex $\langle x^0, \dots, x^d \rangle$, where

$$\lambda_i = \lambda_i(x) = \frac{\text{Vol}_d \langle x^0, \dots, x^{i-1}, x, x^{i+1}, \dots, x^d \rangle}{\text{Vol}_d \langle x^0, \dots, x^d \rangle}.$$

BB-expressions of polynomials and splines-2

Thus, each $\lambda_i = \lambda_i(x)$ is a linear polynomial in x with

$\sum_{i=0}^d \lambda_i = 1$, and if $x \in \langle x^0, \dots, x^d \rangle$, then $\lambda_i \geq 0$.

For any $b = (\beta_0, \dots, \beta_d) \in \mathbb{Z}_+^{d+1}$, and $n \in \mathbb{Z}_+$, we will use the

usual multivariate notation $\lambda^b = \lambda_0^{\beta_0} \cdots \lambda_d^{\beta_d}$, $b! = \beta_0! \cdots \beta_d!$, and

$|b| = \beta_0 + \cdots + \beta_d$. Hence,

$$\phi_b^n(\lambda) := \frac{n!}{b!} \lambda^b \quad (1)$$

is a polynomial in $\pi_{|\beta|}^d$, the space of all polynomials in d variables of order $|\beta| + 1$, or degree at most $|\beta|$.

BB-expressions of polynomials and splines-3

With any set $\{a_\beta^n\} = \{a_\beta^n\}_{\beta \in \mathbb{Z}_+^{d+1}, |\beta|=n} \subset \mathbb{R}$ one may associate the polynomial

$$p_n(x) = B_n[\{a_\beta^n\}; \lambda] = \sum_{|\beta|=n} a_\beta^n \phi_\beta^n(\lambda), \quad (2)$$

which is called a *Bernstein-Bézier polynomial (BB polynomial)* of total degree n relative to the d -simplex $\langle x^0, \dots, x^d \rangle$. In addition, $\{a_\beta^n : |\beta| = n\}$ shown as in (2) is called the set of *Bézier coefficients* of the polynomial p_n . The piecewise linear interpolant to the points $(\beta/n, a_\beta^n)$ is said to be the *Bézier net* or *control net*. and is displayed schematically in Figure 1 for the case of $n = 2$ and $d = 2$.

BB-expressions of polynomials and splines-4

Denote $D_y = \sum_{i=1}^d y_i \frac{\partial}{\partial x_i}$, where $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$. For $y = x^i - x^j$, we denote

$$D_{ij} = D_y = D_{x^i - x^j}, \quad i \neq j.$$

By using the barycentric coordinates $\{\lambda_\ell\}_{\ell=0}^d$ of $x \in \mathbb{R}^d$ relative to an d -simplex $T_d = \langle x^0, \dots, x^d \rangle$, we can write $x = \sum_{\ell=0}^d \lambda_\ell x^\ell$. If we define

$$E_i a_\alpha := a_{\alpha + e^i}$$

and

$$\Delta_{ij} a_\alpha^n = E_i a_\alpha^n - E_j a_\alpha^n,$$

where $e^i = (\delta_{ij})_{j=0}^d$ denotes the i^{th} coordinate vector in \mathbb{R}^{d+1} , then

$$D_{ij} p_n = n \sum_{|\alpha|=n-1} (E_i - E_j) a_\alpha^n \phi_\alpha^{n-1}(\lambda) = n \sum_{|\alpha|=n-1} \Delta_{ij} a_\alpha^n \phi_\alpha^{n-1}(\lambda).$$

(3) 

Continuous Wavelets with Composite Dilation-1

Let B be the group of order 3 generated by the matrix

$$\rho = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

which is the counter-clockwise rotation by

$2\pi/3$. Consider the hexagon H centered at the origin consisting of the diamonds $R_i = (v_{i0}, v_{i1}, v_{i2}, v_{i3})$, ($i = 0, 1, 2$), where v_{i0} , v_{i1} , v_{i2} , and v_{i3} are vertices of R_i , and R_0 has vertices $v_{00} = (0, 0)$, $v_{01} = (\sqrt{3}/4, -3/4)$, $v_{02} = (\sqrt{3}/2, 0)$, $v_{03} = (\sqrt{3}/4, 3/4)$.

Continuous Wavelets with Composite Dilation-2

The elements of B map R_0 onto other diamonds $R_i = \rho^i R_0$ ($i = 1, 2$). Let $C = \frac{1}{4} \begin{pmatrix} 0 & 3\sqrt{3} \\ 6 & 3 \end{pmatrix}$ and $\Gamma_0 = C\mathbb{Z}^2$. The translates of the hexagon by $\gamma \in \Gamma_0$ form a partition of \mathbb{R}^2 with the centers of the hexagons in the partition being the lattice points γ . Let $q = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$. The MRA is now generated by the composite dilation system $\{D_{q^j} D_{\rho^i} T_\gamma : j \in \mathbb{Z}, i = 0, 1, 2, \gamma \in \Gamma_0\}$ applied to the linear scaling function $\phi(x)$ with $\phi(v_{00}) = 1$, $\phi(v_{01}) = \phi(v_{02}) = \phi(v_{03}) = 0$ (i.e., the Bézier coefficient vector of ϕ is $(1, 0, 0, 0)$). Here, the vertex v_{00} at which ϕ has value 1 is the initial vertex of diamond boundary. The space V_j are q^{-j} dilates of V_0 , i.e., $V_j = D_{q^{-j}} V_0$ ($j \in \mathbb{Z}$).

Continuous Wavelets with Composite Dilation-3

The space $V_0 \subset L^2(\mathbb{R}^2)$ consists of the linear functions $\phi_i(x)$ defined on R_i ($i = 0, 1, 2$), with values at vertices of R_i as $\phi(v_{i0}) = 1$ and $\phi(v_{i1}) = \phi(v_{i2}) = \phi(v_{i3}) = 0$, and their translations defined on Γ_0 -translates of the diamonds R_i ($i = 0, 1, 2$). In order to describe the space V_1 we consider the original hexagon H and, within H , the smaller hexagon $q^{-1}H$, which is the disjoint union of the diamonds $R_i = \rho^i R_0$ ($i = 0, 1, 2$) and their translations. $\Phi = [\phi_0, \phi_1, \phi_2]^T$ is refinable. The corresponding multiwavelet Ψ and the duals of the Φ and Ψ are constructed. (More details available upon request.)

C^1 Quadratic Prewavelets with Composite Dilations-1

Consider the hexagonal lattice Δ in \mathbb{R}^2 defined by $C\mathbb{Z}^2$ with $C = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$. Let Δ^3 be the type-3 refinement of Δ . We call $\phi \in S_2^1(\Delta^3)$ a Powell-Sabin(PS) spline or macroelement. For any $k \in \Delta$, the Hermite interpolation problem

$$\begin{pmatrix} \phi_{k,0}(\ell) & D_1\phi_{k,0}(\ell) & D_2\phi_{k,0}(\ell) \\ \phi_{k,1}(\ell) & D_1\phi_{k,1}(\ell) & D_2\phi_{k,1}(\ell) \\ \phi_{k,2}(\ell) & D_1\phi_{k,2}(\ell) & D_2\phi_{k,2}(\ell) \end{pmatrix} = \delta_{k,\ell}$$

has a unique solution $\Phi_k = (\phi_{k,0}, \phi_{k,1}, \phi_{k,2})^T$. And $\{\Phi_{0,k} \equiv \Phi(x - \Gamma k) : k \in \mathbb{Z}^2\}$ is a basis of $S_2^1(\Delta^3)$. The BB-expressions of $\Phi_{0,i}$ ($i = 0, 1, 2$) are given.

C^1 Quadratic Prewavelets with Composite Dilations-2

$\Phi_{0,k}$ is refinable with respect to the dilation matrix $q = 2I$. The refinement $\Delta_j := q^{-j}\Delta$ is the mid-edge subdivision that generates PS partition $\Delta_j^3 := q^{-j}\Delta^3$. The corresponding nested subspaces $V_j = S_2^1(\Delta_j^3) \subset L_2(\mathbb{R}^2)$, $j \in \mathbb{Z}$, form a MRA of multiplicity 3.

- ▶ The MRA generated by applying the composite dilation system $\{D_{q^j} D_{\rho^i} T_\gamma : j \in \mathbb{Z}, i = 0, 1, 2, \gamma \in \Gamma\mathbb{Z}^2\}$ to $\Phi_{0,i}$.
- ▶ Construction of dual of dual basis with composite dilations.
- ▶ Construction of the multi-prewavelets with composite dilations. (More details available upon request.)