Spline Wavelets, Finite Element Wavelets, and Wavelets with Composite Dilation

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Construction of Multivariate Wavelets on Arbitrary Triangulation

- Haar-type non-separable constant wavelets: “twin dragon,” Belogay and Wang [99], Flaherty and Wang [99], and Gröchenig and Madych [92]; wavelet with composite dilations, Krishtal, Robinson, Weiss, and Wilson [08]

- Continuous piecewise linear wavelet: Yserentant [86], Vassilevski and Wang [97], Stevenson [97,98, $H^1$-stable], Liu [06], Floater and Quak [99,00, semi-orthogonal], Hardin and Hong [03, orthogonal on type-1 triangulation]

- $C^1$ quadratic splines and spline wavelets: Powell-Sabin [77], Chui and He[90], Chui, Chui, and He[93], Chui and Jiang [04]; Oswald [92, not $H^2$-stable], Davydov and Petrushev [03,05, $H^\mu$-stable, $\mu \in (1,5/2)$] ($|f|_{H^\mu(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^{2\mu} d\xi\right)^{1/2}$ $< \infty$), Windmolders, Vanraes, Dierckx, and Bultheel [03].
Splines and Elements, Spline Wavelets, Wavelets with Composite Dilations

▶ Splines and their BB-expressions: Farin [88,90,93], Chui [87], etc.

▶ Characterization of compactly supported refinable splines: Lawton, Lee, and Shen [95], Sun [96], Goodman [98], Guan and He [09], etc.

▶ Spline wavelets: Chui and Wang [92,93], Chui, Stöckler, and Ward [92], Jia and Micchelli [91], Riemenschneider and Shen [92], Lorentz and Oswald [00, Sobolev spaces], Jia, Wang, and Zhou [03], Jia and Liu [08], etc.

▶ Wavelets with composite dilations: Guo, Labate, Lim, Weiss, and Wilson [04, 06, 06], etc.
BB-expressions of polynomials and splines-1

Let $x^0, \ldots x^d \in \mathbb{R}^d$, $d \geq 1$, $x^i = (x^i_1, \ldots, x^i_d)$ and consider the convex hull

$$T_d := \langle x^0, \ldots, x^d \rangle = \left\{ \sum_{i=0}^{d} \alpha_i x^i : \sum_{i=0}^{d} \alpha_i = 1, \alpha_i \geq 0 \right\}.$$ 

This convex hull is called an $d$-simplex if its signed volume $Vol_d \langle x^0, \ldots, x^d \rangle$ is nonzero. Suppose that $\langle x^0, \ldots, x^d \rangle$ is an $d$-simplex. Then any $x \in \mathbb{R}^d$ can be identified by an $(d + 1)$-tuple $\lambda = (\lambda_0, \ldots, \lambda_d)$, the barycentric coordinates of $x$ relative to the $d$-simplex $\langle x^0, \ldots, x^d \rangle$, where

$$\lambda_i = \lambda_i(x) = \frac{Vol_d \langle x^0, \ldots, x^{i-1}, x, x^{i+1}, \ldots, x^d \rangle}{Vol_d \langle x^0, \ldots, x^d \rangle}.$$
BB-expressions of polynomials and splines-2

Thus, each $\lambda_i = \lambda_i(x)$ is a linear polynomial in $x$ with $
abla \sum_{i=0}^{d} \lambda_i = 1$, and if $x \in \langle x^0, \ldots, x^d \rangle$, then $\lambda_i \geq 0$.

For any $b = (\beta_0, \ldots, \beta_d) \in \mathbb{Z}_{d+1}^d$, and $n \in \mathbb{Z}_+$, we will use the usual multivariate notation $\lambda^b = \lambda^{\beta_0} \cdots \lambda^{\beta_d}$, $b! = \beta_0! \cdots \beta_d!$, and $|b| = \beta_0 + \cdots + \beta_d$. Hence,

$$\phi^n_b(\lambda) := \frac{n!}{b!} \lambda^b$$ (1)

is a polynomial in $\pi^d_{|\beta|}$, the space of all polynomials in $d$ variables of order $|\beta| + 1$, or degree at most $|\beta|$.
BB-expressions of polynomials and splines-3

With any set \( \{a^n_\beta\} = \{a^n_\beta\}_{\beta \in \mathbb{Z}_+^{d+1}, |\beta| = n} \subset \mathbb{R} \) one may associate the polynomial

\[
p_n(x) = B_n[\{a^n_\beta\}; \lambda] = \sum_{|\beta|=n} a^n_\beta \phi^n_\beta(\lambda), \tag{2}
\]

which is called a \textit{Bernstein-Bézier polynomial (BB polynomial)} of total degree \( n \) relative to the \( d \)-simplex \( \langle x^0, \ldots, x^d \rangle \). In addition, \( \{a^n_\beta : |\beta| = n\} \) shown as in (2) is called the set of \textit{Bézier coefficients} of the polynomial \( p_n \). The piecewise linear interpolant to the points \( (\beta/n, a^n_\beta) \) is said to be the \textit{Bézier net} or \textit{control net}. and is displayed schematically in Figure 1 for the case of \( n = 2 \) and \( d = 2 \).
**BB-expressions of polynomials and splines-4**

Denote

\[ D_y = \sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i}, \]

where \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \). For \( y = x^i - x^j \), we denote

\[ D_{ij} = D_y = D_{x^i - x^j}, \quad i \neq j. \]

By using the barycentric coordinates \( \{\lambda_\ell\}_{\ell=0}^d \) of \( x \in \mathbb{R}^d \) relative to an \( d \)-simplex \( T_d = \langle x^0, \ldots, x^d \rangle \), we can write \( x = \sum_{\ell=0}^{d} \lambda_\ell x^\ell \). If we define

\[ E_i a_\alpha := a_\alpha + e^i \]

and

\[ \triangle_{ij} a^n_\alpha = E_i a^n_\alpha - E_j a^n_\alpha, \]

where \( e^i = (\delta_{ij})_{j=0}^d \) denotes the \( i \)th coordinate vector in \( \mathbb{R}^{d+1} \), then

\[ D_{ij} p_n = n \sum_{|\alpha|=n-1} (E_i - E_j) a^n_\alpha \phi^{n-1}_\alpha(\lambda) = n \sum_{|\alpha|=n-1} \triangle_{ij} a^n_\alpha \phi^{n-1}_\alpha(\lambda). \]

(3)
Continuous Wavelets with Composite Dilation-1

Let $B$ be the group of order 3 generated by the matrix
\[
\rho = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},
\]
which is the counter-clockwise rotation by $2\pi/3$. Consider the hexagon $H$ centered at the origin consisting of the diamonds $R_i = (v_{i0}, v_{i1}, v_{i2}, v_{i3})$, ($i = 0, 1, 2$), where $v_{i0}$, $v_{i1}$, $v_{i2}$, and $v_{i3}$ are vertices of $R_i$, and $R_0$ has vertices $v_{00} = (0, 0)$, $v_{01} = (\sqrt{3}/4, -3/4)$, $v_{02} = (\sqrt{3}/2, 0)$, $v_{03} = (\sqrt{3}/4, 3/4)$. 
Continuous Wavelets with Composite Dilation-2

The elements of $B$ map $R_0$ onto other diamonds $R_i = \rho^i R_0$ ($i = 1, 2$). Let $C = \frac{1}{4} \begin{pmatrix} 0 & 3\sqrt{3} \\ 6 & 3 \end{pmatrix}$ and $\Gamma_0 = C\mathbb{Z}^2$. The translates of the hexagon by $\gamma \in \Gamma_0$ form a partition of $\mathbb{R}^2$ with the centers of the hexagons in the partition being the lattice points $\gamma$. Let $q = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$. The MRA is now generated by the composite dilation system $\{D_q^j D_{\rho^i} T_{\gamma} : j \in \mathbb{Z}, i = 0, 1, 2, \gamma \in \Gamma_0\}$ applied to the linear scaling function $\phi(x)$ with $\phi(v_{00}) = 1$, $\phi(v_{01}) = \phi(v_{02}) = \phi(v_{03}) = 0$ (i.e., the Bézier coefficient vector of $\phi$ is $(1, 0, 0, 0)$). Here, the vertex $v_{00}$ at which $\phi$ has value 1 is the initial vertex of diamond boundary. The space $V_j$ are $q^{-j}$ dilates of $V_0$, i.e., $V_j = D_{q^{-j}} V_0$ ($j \in \mathbb{Z}$).
Continuous Wavelets with Composite Dilation-3

The space $V_0 \subset L^2(\mathbb{R}^2)$ consists of the linear functions $\phi_i(x)$ defined on $R_i$ ($i = 0, 1, 2$), with values at vertices of $R_i$ as $\phi(v_{i0}) = 1$ and $\phi(v_{i1}) = \phi(v_{i2}) = \phi(v_{i3}) = 0$, and their translations defined on $\Gamma_0$-translates of the diamonds $R_i$ ($i = 0, 1, 2$). In order to describe the space $V_1$ we consider the original hexagon $H$ and, within $H$, the smaller hexagon $q^{-1}H$, which is the disjoint union of the diamonds $R_i = \rho^i R_0$ ($i = 0, 1, 2$) and their translations. $\Phi = [\phi_0, \phi_1, \phi_2]^T$ is refinable. The corresponding multiwavelet $\Psi$ and the duals of the $\Phi$ and $\Psi$ are constructed. (More details available upon request.)
Consider the hexagonal lattice \( \Delta \) in \( \mathbb{R}^2 \) defined by \( C\mathbb{Z}^2 \) with \( C = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} \). Let \( \Delta^3 \) be the type-3 refinement of \( \Delta \). We call \( \phi \in S_2^1(\Delta^3) \) a Powell-Sabin(PS) spline or macroelement. For any \( k \in \Delta \), the Hermite interpolation problem

\[
\begin{pmatrix}
\phi_{k,0}(\ell) & D_1\phi_{k,0}(\ell) & D_2\phi_{k,0}(\ell) \\
\phi_{k,1}(\ell) & D_1\phi_{k,1}(\ell) & D_2\phi_{k,1}(\ell) \\
\phi_{k,2}(\ell) & D_1\phi_{k,2}(\ell) & D_2\phi_{k,2}(\ell)
\end{pmatrix} = \delta_{k,\ell}I
\]

has a unique solution \( \Phi_k = (\phi_{k,0}, \phi_{k,1}, \phi_{k,2})^T \). And \( \{\Phi_{0,k} \equiv \Phi(x - \Gamma k) : k \in \mathbb{Z}^2\} \) is a basis of \( S_2^1(\Delta^3) \). The BB-expressions of \( \Phi_{0,i} \) (\( i = 0, 1, 2 \)) are given.
$C^1$ Quadratic Prewavelets with Composite Dilations-2

$\Phi_{0,k}$ is refinable with respect to the dilation matrix $q = 2I$. The refinement $\Delta_j := q^{-j}\Delta$ is the mid-edge subdivision that generates PS partition $\Delta_j^3 := q^{-j}\Delta^3$. The corresponding nested subspaces $V_j = S_2^1(\Delta_j^3) \subset L_2(\mathbb{R}^2), j \in \mathbb{Z}$, form a MRA of multiplicity 3.

- The MRA generated by applying the composite dilation system $\{D_q D_{\rho_i} T_\gamma : j \in \mathbb{Z}, i = 0, 1, 2, \gamma \in \Gamma \mathbb{Z}^2\}$ to $\Phi_{0,i}$.
- Construction of dual of dual basis with composite dilations.
- Construction of the multi-prewavelets with composite dilations. (More details available upon request.)