Biorthogonal Wavelets with 6-fold Symmetry
for Hexagonal Data and Triangle Surface
Multiresolution Processing

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Qingtang Jiang
Dept. of Math and Computer Sci.
University of Missouri-St. Louis
St. Louis, MO 63121, U.S.A.
Outline of the Talk

- Hexagonal sampling
- Triangle surface subdivision
- Biorthogonal wavelets with 6-fold symmetry
- Lifting-scheme-based multiresolution alg.
1 Hexagonal sampling

Figure 1: Left: Rectangular lattice (array); Right: Hexagonal lattice (array)

Figure 2: Left: Square tessellation; Right: Hexagonal tessellation
Hexagonal lattice

- Fewer sampling points to maintain equally high frequency information
- 6 directional symmetry (while 4-fold symmetry for square lattice)
- Better consistent connectivity
- Orientation banding
- ...
- Pertinent to the vision process

applied in edge detection, pattern recognition, ...
used in Geoscience
\text{supp} \ (\hat{f}) \subset \{ \omega \in \mathbb{R}^2 : |\omega| \leq R \}

**Fewer hexagonal samples** needed to recover \( f \).

SMOS (Soil Moisture and Ocean Salinity) mission by European Space Agency

Figure 3: Y-shaped antenna picks up hexagonally sampled data (picture from Eesa)
Figure 4: Top: Dyadic refinement; Bottom: $\sqrt{3}$ and $\sqrt{7}$ refinements

Highest symmetry may a filter bank $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$ have?
2 Triangle surface subdivision

Subdivision: coarse mesh $\implies$ finer mesh

Figure 5: Subdivision produces smooth surface (picture from Zorin & Schroeder SIGGRAPH 99 Course Notes)

Surface multiresolution processing:

\begin{align*}
\text{decomp.} & \quad \text{coarse mesh} & \quad \text{finer (original) mesh} \\
\downarrow & \quad \uparrow \quad \text{details}
\end{align*}
Regular mesh: (The valence of each vertex is 6).

Figure 6: Left: Regular mesh; Right: Extraordinary vertex E

$e_1, e_2, e_3$ have the same algorithm
$V_1, V_2, V_3$ have the identical algorithm

Rotation and reflection invariant algs.
3 6-fold symmetric biorthogonal wavelets

Figure 7: Left: Indices for hexagonal nodes; Right: Indices for hexagonally sampled data $\mathcal{C}$

Decomposition alg. with \{\(p, q^{(1)}, q^{(2)}, q^{(3)}\)\}:

\[
c_{j+1}^{n} = \frac{1}{4} \sum_{k \in \mathbb{Z}^2} p_{k-2n} c_{k}^{j}, \quad d_{n}^{(\ell,j+1)} = \frac{1}{4} \sum_{k \in \mathbb{Z}^2} q_{k-2n}^{(\ell)} d_{k}^{j},
\]

Reconstruction alg. with \{\(\tilde{p}, \tilde{q}^{(1)}, \tilde{q}^{(2)}, \tilde{q}^{(3)}\)\}:

\[
c_{k}^{j} = \sum_{n \in \mathbb{Z}^2} \tilde{p}_{k-2n} c_{n}^{j+1} + \sum_{1 \leq \ell \leq 3} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-2n}^{(\ell)} d_{n}^{(\ell,j+1)}
\]
Filter bank $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$ is said to have 6-fold axial (line) symmetry if

- (i) lowpass filter $p(\omega)$ is symmetric around $S_0, \ldots, S_5$,
- (ii) $e^{-i(\omega_1 + \omega_2)}q^{(1)}(\omega)$ is symmetric around the axes $S_0, S_3$, and
- (iii) $q^{(2)}$ and $q^{(3)}$ are the $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ rotations of $q^{(1)}$ resp.
For \(\{p, q^{(1)}, q^{(2)}, q^{(3)}\}\), with \(q^{(0)} = p\), write \(q^{(\ell)}(\omega)\):

\[
q^{(\ell)}(\omega) = \frac{1}{2}(q_0^{(\ell)}(2\omega) + q_1^{(\ell)}(2\omega)e^{i(\omega_1 + \omega_2)} + q_2^{(\ell)}(2\omega)e^{-i\omega_1} + q_3^{(\ell)}(2\omega)e^{-i\omega_2}),
\]

\(q_k^{(\ell)}\)-trigonometric poly. Polyphase matrix \(V(\omega)\):

\[
V(\omega) = \left[q_k^{(\ell)}(\omega)\right]_{0 \leq \ell, k \leq 3};
\]

\[\left[p, q^{(1)}, q^{(2)}, q^{(3)}\right]^T(\omega) = \frac{1}{2}V(2\omega)I_{00}(\omega),\]

\(I_{00}(\omega) = [1, e^{i(\omega_1 + \omega_2)}, e^{-i\omega_1}, e^{-i\omega_2}]^T\).
Prop. 1 \{p, q^{(1)}, q^{(2)}, q^{(3)}\} has 6-fold axial symmetry iff

\[
V(R_1^{-T}\omega) = S_1(\omega)V(\omega)S_2(\omega), \\
V(L_0\omega) = S_0V(\omega)S_0,
\]

\[
S_1(\omega) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{i\omega_2} \\
0 & e^{-i(\omega_1+\omega_2)} & 0 & 0 \\
0 & 0 & e^{i\omega_1} & 0
\end{pmatrix}, \\
S_2(\omega) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & e^{i(\omega_1+\omega_2)} & 0 \\
0 & 0 & 0 & e^{-i\omega_1} \\
0 & e^{-i\omega_2} & 0 & 0
\end{pmatrix}, \\
S_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
L_0 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \\
R_1 = \begin{pmatrix}
0 & 1 \\
-1 & 1
\end{pmatrix}.
\]
Prop. 2 Suppose \( \{p, q^{(1)}, q^{(2)}, q^{(3)}\} \) and \( \{\tilde{p}, \tilde{q}^{(1)}, \tilde{q}^{(2)}, \tilde{q}^{(3)}\} \) are given by

\[
[p, q^{(1)}, q^{(2)}, q^{(3)}]^T(\omega) = V_n(2\omega)V_{n-1}(2\omega) \cdots V_0(2\omega)I_{00}(\omega),
\]

\[
[\tilde{p}, \tilde{q}^{(1)}, \tilde{q}^{(2)}, \tilde{q}^{(3)}]^T(\omega) = \frac{1}{4} V_n(2\omega)V_{n-1}(2\omega) \cdots \tilde{V}_0(2\omega)I_{00}(\omega).
\]

Then \( \{p, q^{(1)}, q^{(2)}, q^{(3)}\} \) and \( \{\tilde{p}, \tilde{q}^{(1)}, \tilde{q}^{(2)}, \tilde{q}^{(3)}\} \) are biorthogonal FIR filter banks with 6-fold symmetry, where \( \tilde{V}_j(\omega) = (V_j(\omega)^{-1})^* \),

with

\[
x = e^{-i\omega_1}, \quad y = e^{-i\omega_2},
\]

\[
V_j(\omega) = \begin{bmatrix}
\gamma + \rho(x + xy + y + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}) & \alpha(1 + x + y) + \beta(1 + \frac{1}{x} + \frac{1}{y}) & \alpha(1 + \frac{1}{x}) + \beta(1 + \frac{1}{y}) \\
\tau(1 + x + y) & 0 & 0 \\
\tau(1 + x) & 0 & 1 \\
\tau(1 + y) & 1 & 0
\end{bmatrix},
\]

\[
\tilde{V}_j(\omega) = \frac{1}{\tau(1 + x + y)} \begin{bmatrix}
-\tau(1 + x + y) & -\tau(1 + x) & -\tau(1 + y) \\
-\tau(x + y + \frac{1}{x}) & \xi(x, y) & \tau(1 + x + \frac{1}{x}) \left( \alpha + \alpha x + \beta xy + \frac{\beta}{x} \right) \\
-\tau(x + y + \frac{1}{y}) & \tau(1 + x + \frac{1}{x}) \left( \alpha + \alpha y + \beta xy + \frac{\beta}{y} \right) & \xi(y, x)
\end{bmatrix}.
\]

6-fold axial symmetric filter banks yield

algs. with required symmetry for surface multiresolution proc.
4 Lifting-scheme-based multiresolution algorithms

Lifting-scheme-based multiresolution alg. was introduced:


works for regular and irregular meshes. But no filter banks, no properties of $\psi^f, \phi^f$.

Figure 9: Left: Regular triangle mesh; Right: Initial data separated into 4 groups: $\{v_k\}, \{e_k^{(1)}\}, \{e_k^{(2)}\}, \{e_k^{(3)}\}$

$v_k = c_{2k}, e_k^{(1)} = c_{2k-1,1}, e_k^{(2)} = c_{2k+1,0}, e_k^{(3)} = c_{2k+0,1}, k \in \mathbb{Z}^2.$
Decomposed data:
\{c^1_k\} – “approx. part”, \{d^{(\ell,1)}_k\}, \ell = 1, 2, 3 – “details”.

Denote
\[ \tilde{v}_k = c^1_k, \tilde{e}^{(1)}_k = d^{(1,1)}_k, \tilde{e}^{(2)}_k = d^{(2,1)}_k, \tilde{e}^{(3)}_k = d^{(3,1)}_k. \]

Associated \( \tilde{v}_k, \tilde{e}^{(1)}_k, \tilde{e}^{(2)}_k, \tilde{e}^{(3)}_k \) with
\[ 2k, 2k - (1,1), 2k + (1,0), 2k + (0,1). \]

Figure 10: Decomposed data associated with 4 groups of hexagonal nodes
Since the same alg. for \( e_k^{(1)} \), \( e_k^{(2)} \), \( e_k^{(3)} \), and the same alg. for \( \tilde{e}_k^{(1)} \), \( \tilde{e}_k^{(2)} \), \( \tilde{e}_k^{(3)} \),

use \( e \) to denote \( e_k^{(1)} \), \( e_k^{(2)} \), \( e_k^{(3)} \), and \( \tilde{e} \) denotes \( \tilde{e}_k^{(1)} \), \( \tilde{e}_k^{(2)} \), \( \tilde{e}_k^{(3)} \).

Figure 11: Decomposition and reconstruction algorithms
Decomposition Algorithm:
Step 1. $v'' = \frac{1}{b}\{v - d(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)\}$
Step 2. $e'' = e - a(v''_0 + v''_1) - c(v''_2 + v''_3)$
Step 3. $\tilde{v} = v'' - w(e''_0 + e''_1 + e''_2 + e''_3 + e''_4 + e''_5)$
$\quad - u(e''_6 + e''_7 + e''_8 + e''_9 + e''_{10} + e''_{11})$
Step 4. $\tilde{e} = e'' - s(\tilde{v}_0 + \tilde{v}_1) - r(\tilde{v}_2 + \tilde{v}_3)$. 
Figure 12: Top-left: Decomposition Alg. Step 1; Top-right: Decomposition Alg. Step 2; Bottom-left: Decomposition Alg. Step 3; Bottom-right: Decomposition Alg. Step 4
Reconstruction Algorithm:

Step 1. \( e'' = \tilde{e} + s(\tilde{v}_0 + \tilde{v}_1) + r(\tilde{v}_2 + \tilde{v}_3) \)

Step 2. \( v'' = \tilde{v} + w(e''_0 + e''_1 + e''_2 + e''_3 + e''_4 + e''_5) \)

\[ + u(e''_6 + e''_7 + e''_8 + e''_9 + e''_{10} + e''_{11}) \]

Step 3. \( e = e'' + a(v''_0 + v''_1) + c(v''_2 + v''_3) \)

Step 4. \( v = bv'' + d(e_0 + e_1 + e_2 + e_3 + e_4 + e_5) \).
Figure 13: Top-left: Reconstruction Alg. Step 1; Top-right: Reconstruction Alg. Step 2; Bottom-left: Reconstruction Alg. Step 3; Bottom-right: Reconstruction Alg. Step 4
Thanks for your attention!