Time-Frequency localization of Multiband signals
AMS Meeting # 1047, Urbana, March 27, 2009

Joe Lakey (w Scott Izu)¹

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Fourier transform: 
\[ \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \]
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\( P_\Sigma f(x) = (\hat{f} 1_\Sigma)^\vee(x) \); Paley-Wiener: \( PW_\Sigma = P_\Sigma(L^2(\mathbb{R})) \)
Time and frequency localization

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Fundamental Questions Include . . .:

- Time localization of \( \text{PW}_\Sigma \)
  - \( Q_s f(x) = f(x) \mathbb{1}_s(x); \)

Especially, \( s \) an interval
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt \)

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- Time localization of \( \text{PW}_{\Sigma} \)
- \( Q_{S} f(x) = f(x) \, \mathbb{1}_{S}(x) \);
- Eigenvalues of \( P_{\Sigma} Q_{S} P_{\Sigma} \): vs \( |S| \, |\Sigma| \), linear distribution of \( \Sigma \)
**Time and frequency localization**

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  - Sampling theory of \( \text{PW}_\Sigma \)
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  - Sampling theory of \( PW_{\Sigma} \)
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    - \( \phi_n \) eigenvectors of \( P_{\Sigma} QS \),
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Fundamental Questions Include . . . :

- Time localization of $PW_\Sigma$
  - $Q_\Sigma f(x) = f(x) \mathbb{1}_S(x)$;
  - Eigenvalues of $P_\Sigma Q_\Sigma P_\Sigma$: vs $|S||\Sigma|$, linear distribution of $\Sigma$
  - Especially, $S$ an interval

- Sampling theory of $PW_\Sigma$
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  - $\phi_n$ eigenvectors of $P_\Sigma Q_\Sigma$,
  - Quantify $\langle f, \phi_n \rangle$ in terms of $\{f(x_k)\}$
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  - . . . finite-dimensional approximations
Time and frequency localization

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    - . . . finite-dimensional approximations
  - FFT version . . . communications applications
I. Time and frequency localization: Bell Labs Theory

\[ P \Sigma Q S P \text{ self-adjoint}, \]
\[ \lambda_{\text{max}} = \lambda_0 = \| P \Sigma Q S \| = \sup_{f \in \mathcal{P}, \| f \| = 1} \| Q S(f) \|_2 \]

Uncertainty principle:
\[ \lambda_{\text{max}} < 1 \text{ if } |S| |\Sigma| < \infty \]

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I. Time and frequency localization: Bell Labs Theory

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- Uncertainty principle: $\lambda_{\text{max}} < 1$ if $|S||\Sigma| < \infty$
\begin{itemize}
  \item \( S = [-T/2, T/2] \); \( \Sigma = [-\Omega/2, \Omega/2] \), \( \operatorname{tr} P_\Omega Q_T = T\Omega \equiv c \).
\end{itemize}
- $S = [-T/2, T/2]$; $\Sigma = [-\Omega/2, \Omega/2]$, $\text{tr } P_\Omega Q_T = T\Omega \equiv c$.
- Orthonormal eigenfunctions: $P_\Omega Q_T \phi_j = \lambda_j \phi_j$
Prolate spheroidal wave functions

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- $P_\Omega Q_T$ commutes with

$$\left(T^2 - t^2\right) \frac{d^2}{dt^2} - 2t \frac{d}{dt} - \Omega^2 t^2$$
Prolate spheroidal wave functions

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  \[
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  \]
- Eigenfunctions are Prolate Spheroidal Wave Functions
Approximately $c = \Omega T$ eigenvalues close to one
Eigenvalue properties

- Approximately $c = \Omega^T$ eigenvalues close to one
- Plunge region of width $\approx \log c$
Eigenvalue properties

- Approximately $c = \Omega T$ eigenvalues close to one
- Plunge region of width $\approx \log c$
- Transition about $j = [c]$: $\lambda_{[c]+1} \leq 1/2 \leq \lambda_{[c]-1}$
**Figure:** *Eigenvalues for one frequency interval.* \( N = 1025 \) point centered DFT. \( S \sim 513 + [-128, 128] \); \( \Sigma \sim 513 + [-128, 128] \).

\( c = \#T \times \#\Sigma/N \approx 64 \). Plunge region \( \sim 61 \leq n \leq 69 \).
Figure: Even eigenvectors for one frequency interval. $N = 129$ point centered DFT. $S \sim 65 + [-16, 16]$; $\Sigma \sim 65 + [-16, 16]$. $c = \#T \times \#\Sigma/N = 8.44$. Plunge region $\sim 7 \leq n \leq 12$. 
**Figure:** *Eigenvalues for two frequency intervals.* $N = 1025$ point centered DFT. $S \sim 513 + [-128, 128]$; $\Sigma \sim 512 + [-64, 64] \cup [128, 192]$. $c \approx \#T \times \#\Sigma/N \approx 64$. Plunge region $\sim 55 \leq n \leq 84$. 
Figure: Discrete eigenvectors, two “symmetric” frequency intervals.
Normalized area 24.69
“RANDOM” frequency support

Figure: Random $\Sigma$, $N = 129$. $c = 24.69$. 
Figure: *Eigenvalues for random* $\Sigma$. $N = 129$, $c = 24.69$. Flat around $c = |\Sigma|$
Figure: Eigenvectors (real parts) for random $\Sigma$. $N = 129$, $c = 24.69$
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

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- $A_c = P_{c\Sigma}QS P_{c\Sigma}$
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- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
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- $N(A_c, \alpha) = \#\{\lambda(A_c) > \alpha\}$
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The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

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- Area $c|S||\Sigma|$ for $\alpha = 1/2$ in limit.
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

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- Area $c|S||\Sigma|$ for $\alpha = 1/2$ in limit.
- $N_S N_\Sigma$: width of “plunge region”
Plunge width $\sim N_s N_\Sigma$
Plunge width $\sim N_\Sigma N_\Sigma$

- Separated at infinity
Plunge width $\sim N_\Sigma N_\Sigma$

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- $\phi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
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- $\phi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
- $\phi_j(t) = e^{2\pi im_j t}\varphi_j(t)$ $m_j = I_j$. 

Each $I_j$ gives one eigenvalue $\approx 1/2$.

If each $I_j$ were very short: no large eigenvalues.
Plunge width $\sim N_S N_\Sigma$

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\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \overline{\varphi_k(t)} \, dt
\]

\[
= \hat{\varphi}_1 \ast \hat{\varphi}_2 \ast \text{sinc} (m_1 - m_2) = O(1/|m_1 - m_2|)
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- $\phi_j$ frequency concentrated on $I_j$, $|I_j| = 1$
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- Each $l_j$ gives one eigenvalue $\approx 1/2$
- If each $l_j$ were very short: no large eigenvalues.
More discrete illustrations: random frequencies
Structured Fourier spectrum (DFT!)

Play/PauseSlow
When is area formula $\lambda_{[c]} \geq 1/2$ still valid?

**Proposition**

(Izu)

Let $\Sigma = [-1/2, 1/2]$ and let $S$ be a union of $m$ pairwise disjoint intervals of total length $c$. Set

$$\nu = \max_{\alpha} \# \{ k \in \mathbb{Z} : (k, k + 1) \subset S + \alpha \},$$

$$\mu = \min_{\beta} \# \{ \ell \in \mathbb{Z} : (\ell, \ell + 1) \cap S + \beta \neq \emptyset \}.$$

Then the eigenvalues $\lambda_k$ of $Q_S P$ satisfy

$$\lambda_{\nu - 1} \geq 1/2 \geq \lambda_{\mu}.$$
Corollary
When $T = 1$ and $\Sigma$ is a union of integer intervals $[k, k + 1]$, $\lambda_c = 1/2$.

Conjecture
When $\Sigma$ is a symmetric union of “grid intervals” of length $1/T$ (so $c \in \mathbb{N}$) one has $\lambda_{c-k} + \lambda_{c+k} = 1$, $k = 1, \ldots, [c]$. 
Figure: Eigenvalues for DFT localization, $N = 1024$, $T = 128$, 10 symmetrized length 16 intervals $c = 40$ (real part), Note symmetry
Figure: Σ: 10 symmetrized length 16 intervals $c = 40$ (real part)
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then ...
Largest energy concentration for a given area?

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\[
\int_{-T/2}^{T/2} |f(t)|^2 dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)^\vee(t)|^2 dt.
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$$\int_{-T/2}^{T/2} |f(t)|^2 dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)^\vee(t)|^2 dt.$$  

- Optimal concentration: $\Sigma$ is an interval if $T$ is small enough.
Largest energy concentration for a given area?

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- Optimal concentration: $\Sigma$ is an interval if $T$ is small enough.

- Rearrangement inequality fails for large measure.
\( (\mathcal{S}, \Sigma) \) supports information if \( \|P_\Sigma Q_\mathcal{S} P_\Sigma\| \geq 1/2 \).
(S, Σ) supports information if $\|P_\Sigma Q S P_\Sigma\| \geq 1/2$.

... at rate $N$: $N$ eigenvalues $\geq 1/2$
Information problem

- $(S, \Sigma)$ supports information if $\|P_\Sigma Q_S P_\Sigma\| \geq 1/2$.
- ... at rate $N$: $N$ eigenvalues $\geq 1/2$
- Rationale: basis functions $\sim$ codes
Information problem

- $(S, \Sigma)$ supports information if $\|P_\Sigma Q_S P_\Sigma\| \geq 1/2$.
- ... at rate $N$: $N$ eigenvalues $\geq 1/2$
- Rationale: basis functions $\sim$ codes
- Which pairs support information?
Theorem

(Candès, Romberg, Tao) Fix \( N \geq 512 \) and \( \beta \) such that \( 1 \leq \beta \leq (3/8) \log N \). Suppose that \( S \) and \( \Sigma \) are subsets of \( \mathbb{Z}_N \) whose sizes satisfy

\[
|S| + |\Sigma| \leq M(N, \beta) = \frac{N}{\sqrt{\beta + 1} \log N} \left( \frac{1}{\sqrt{6}} + o(1) \right).
\]

Then with probability at least \( 1 - O((\log N)^{1/2} / N^\beta) \), every signal \( x \) frequency supported in \( \Sigma \) satisfies

\[
\|x 1_S\|^2 \leq \frac{1}{2} \|x\|^2.
\]
The entropy of a partition $\mathcal{P}$ of a probability space $(X, \mathcal{B}, \mu)$ is 

$$E(P) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

**Problem**

Let $S \sim [-T/2, T/2]$ and let $|\Sigma|$ and hence $|S||\Sigma|$ be fixed in the finite time-frequency plane. Establish a quantitative, probabilistic relationship between an appropriate entropy of $\Sigma$ and an appropriate norm of $A_{S\Sigma}$. 
Figure: Norm of $P_{\Sigma}Q_T$ versus entropy, $N = 512$, $c = 8$
IV. Sampling and Time-Frequency localization
Sampling and eigenfunctions: $\Omega_T$ case

Theorem
(Shen and Walter; Khare and George)

$\varphi_n \sim \lambda_n$ of $PQ_T P$. Then

\[
\lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k)
\]

where the doubly-infinite matrix $A$ has entries $A_{mk}$ given by

$$A_{mk} = \int_{-T/2}^{T/2} \text{sinc}(t-m) \text{sinc}(t-k) \, dt.$$
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where the doubly-infinite matrix \(A\) has entries \(A_{mk}\) given by

\[A_{mk} = \int_{-T/2}^{T/2} \text{sinc}(t - m) \text{sinc}(t - k) \, dt.\]

i.e. samples of \(\varphi_n\) form \(n\)-th eigenvector of \(\{A_{mk}\}\).
Problem

Quantify the sense in which the eigenvectors of the matrix \( \tilde{A} \) obtained by truncating \( A_{mk} \) to zero where \( \max\{m, k\} > N \) approximate those of \( A \).
\[ A_{k\ell} = \int_{-T}^{T} \text{sinc} (x - k) \text{sinc} (x - \ell) \, dx. \]

**Proposition**

(Izu, L.)

(i) When \( \ell > k \geq T \):

\[ A_{k\ell} = \frac{(-1)^{k-\ell}}{\pi} \frac{2T}{(k + T)(\ell + T)} + O\left(\frac{1}{k^2(\ell - k)}\right), \quad \text{as } k, \ell \to \infty. \]

(ii) Let \( A_{k\ell}^{\text{trunc}} = A_{k\ell} \) if \( \max\{|k|,|\ell|\} \leq NT \) and \( A_{k\ell}^{\text{trunc}} = 0 \) otherwise. Set \( \tilde{A} = A - A_{k\ell}^{\text{trunc}}. \) Then \( \|\tilde{A}\|_{\ell^2 \to \ell^2} \approx C(NT)^{-1/2} \)

where \( C \) is a fixed constant independent of \( N \) and \( T \).
Problem
Quantify the sense in which the eigenvectors of the matrix \( \tilde{A} \) obtained by truncating \( A_{mk} \) to zero where \( \max\{m, k\} > N \) approximate those of \( A \).
Shen and Walter: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on area
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Joe Lakey (w Scott Izu)  |  Time-frequency multiband
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- **Generalizability** to multiband?
Figure: Interpolation from truncation approximations. $T = 5, N = 10$: 21 terms (left). DPSS sequences $[E,V]=\text{dpss}(120,10)$ (right). Approximation is excellent for PSWFs with $\lambda \approx 1$. 
Problem

Describe projection onto localized eigenspaces of $P_{\Sigma} Q_T P_{\Sigma}$ in terms of samples of eigenvectors in the multiband case.
First steps: Sampling and eigenfunctions

- $S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n$: $\forall f \in PW_\Sigma, f(t) = \sum_n f(x_n)\psi_n(t)$
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- Then

\[
P_{\Sigma}Q_Sf(x_n) = \int_S \left( \sum_m f(x_n)\psi_m(t) \right) g_n(t) \, dt = \sum_m B_{nm}f(x_m)
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- and

$$P_\Sigma Q_S f(t) = \sum_n (P_\Sigma Q_S f)(x_n)\psi_n(t) = \sum_n \left(\sum_m B_{nm} f(x_n)\right)\psi_n(t)$$
Theorem

(Izu) If $\varphi$ is a $\lambda$-eigenfunction of $P_\Sigma Q_S$ then $\{\varphi(x_n)\}$ is a $\lambda$-eigenvector of $B$. Conversely, if $v$ is a $\lambda$-eigenvector of $B$ and if $\varphi(t) = \sum_m v_m g_m(t)$ converges then $\varphi$ is a $\lambda$-eigenfunction of $P_\Sigma Q_S$. 
Next steps . . .: Sampling of multiband signals

- Venkataramani and Bresler: periodic nonuniform sampling; Interpolating functions . . .
Next steps . . .: Sampling of multiband signals

► Venkataramani and Bresler: periodic nonuniform sampling; Interpolating functions . . .
► Many other approaches: Herley and Wong, Behmard Faridani and Walnut, Avdonin and Moran . . .
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Summary

- In multiband case still need
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Applications to ... communications ...