Frequency-scale frames and the solution of the Mexican hat problem

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Every "nice" ON wavelet basis for L^2 is an unconditional basis for $L^p, 1 < p < \infty$

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No! For each $p > 2$, there exists a "nice" Riesz wavelet basis of L^2 that does not even *span* L^p. (Tchamitchian, Lemarié)

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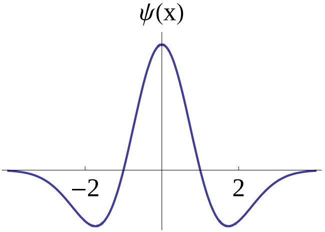
Chastened, we seek *sufficient* conditions under which L^2 wavelet properties *do* persist to all L^p

Mexican hat model problem

Mexican hat
$\psi(x) = -\left(e^{-x^2/2}\right)'' = (1 - x^2)e^{-x^2/2}$



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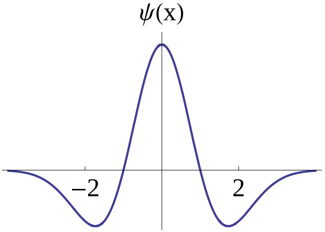

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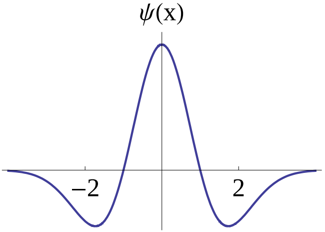

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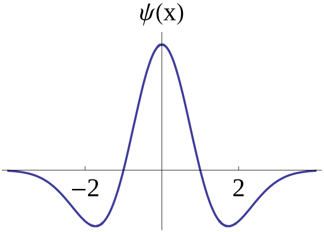

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Endpoint answers:
no for L^∞; no for L^1 (since $\psi_{j,k}$ integrates to zero); plausible for H^1

Time-scale (wavelet) frame notation

Translates and dilates

$$\psi_{j,k}(x) = 2^{j/p}\psi(2^j x - k) \qquad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$
$$\phi_{j,k}(x) = 2^{j/q}\phi(2^j x - k)$$

Define

$$\text{synthesis operator} \quad s(\{c_{j,k}\}) = \sum c_{j,k}\psi_{j,k}$$
$$\text{analysis operator} \quad t(f) = \{\langle f, \phi_{j,k}\rangle\}$$
$$\text{mixed frame operator} \quad (s \circ t)(f) = \sum \langle f, \phi_{j,k}\rangle \psi_{j,k}$$

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So seek **approximately dual** frame analyzer (Christensen & Laugesen 2008):

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Still too difficult in $L^p, p \neq 2$, due to oscillations in ψ, ϕ.
So move to **frequency domain**...

Frequency-scale frame notation Define modulates and dilates in frequency domain:

$$\Phi = \mathcal{F}^{-1}\phi, \quad \Phi_{j,k} = \mathcal{F}^{-1}\phi_{j,k} \qquad \Psi = \mathcal{F}^{-1}\psi, \quad \Psi_{j,k} = \mathcal{F}^{-1}\psi_{j,k}$$

$$\begin{array}{ccc} F & \xrightarrow{T} & \{C_j\} \\ \mathcal{F}\downarrow & & \downarrow\mathcal{F} \\ f & \xrightarrow[t]{} & \{c_{j,k}\} \end{array} \qquad \begin{array}{ccc} \{C_j\} & \xrightarrow{S} & F \\ \mathcal{F}\downarrow & & \downarrow\mathcal{F} \\ \{c_{j,k}\} & \xrightarrow[s]{} & f \end{array}$$

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Commutative diagrams for frequency-scale synthesis S, analysis T, and time-scale synthesis s, analysis t, via Fourier transform $\mathcal{F}$.
Here $C_j(\xi) = \sum_{k\in\mathbb{Z}} c_{j,k} e^{2\pi i \xi k}$.

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Function spaces

$$S : \ell^q\big(L^q(\mathbb{T})\big) \to L^q(\mathbb{R})$$
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are bounded if Ψ, Φ decay near 0 and ∞ and $\boxed{1 \leq q < \infty}$
Littlewood–Paley not needed, because the periodic function C_j is normed, not its Fourier coefficients.

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Given synthesizer Ψ, want analyzer Φ giving approximate reconstruction:

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Theorem (Frequency-scale frame reconstruction estimate)

The Calderón condition $\sum_j \Psi(\xi 2^j)\overline{\Phi(\xi 2^j)} \equiv 1$ *implies*

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$$\leq \Delta(\Phi, \Psi) = \sum_{l \neq 0} \| \sum_j |\Phi(\xi 2^j)\Psi(\xi 2^j - l)| \|_\infty^{1/q} \cdot \| \sum_j |\Phi(\xi 2^j + l)\Psi(\xi 2^j)| \|_\infty^{1/p}$$

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Proof. Adapt Daubechies' L^2 method to our function spaces in frequency domain. Terms with $l = 0$ yield identity I, by Calderón.

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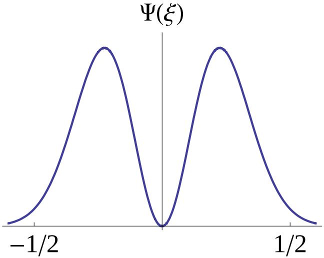
Corollary (Frequency-scale frames, time-scale spanning)

If $\Delta(\Phi, \Psi) < 1$ *then* $S \circ T$ *is bijection, hence* $\{\Psi_{j,k}\}$ *spans* L^q*. So by Hausdorff–Young, if* $1 < q \leq 2$ *(*$2 \leq p < \infty$*), then* $\{\psi_{j,k}\}$ *spans* L^p*.*

Mexican hat example — constructing good analyzer Φ



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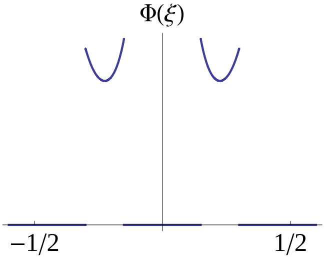

$\Phi = \mathbb{1}_E / \Psi$ where
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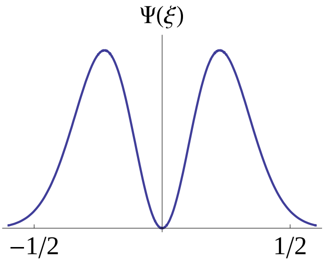

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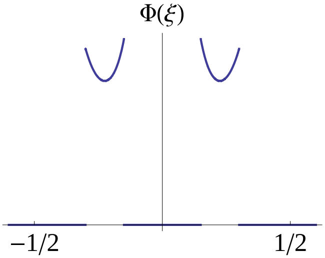

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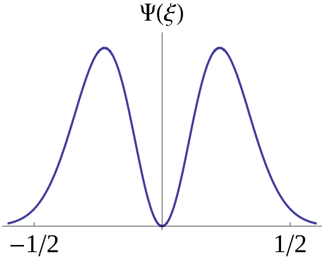

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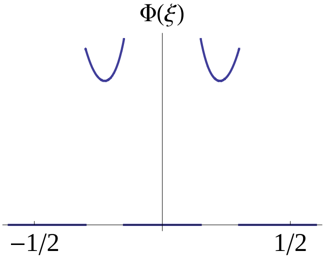

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We prove $\Delta(\Phi, \Psi) \simeq 10^{-4} < 1$. So reconstruction is almost perfect. Corollary implies:

Mexican hat spanning problem solved for $2 \leq p < \infty$

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for approximate time-scale reconstruction $\|s \circ t - \text{id.}\| < 1$, in this weighted space.

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• Application: verify sufficient condition for Mexican hat system. Conclude it spans $L^p, 1 < p \leq 2$ and Hardy space $H^p, 2/3 < p \leq 1$.

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Future directions

Spanning in the time domain is good.

Bijectivity of the time-scale frame operator $s \circ t$ would be better, giving *reconstruction* formulas in time domain.