Notions of equivalence of generalized multiresolution analyses

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Background:

1999 – L. Baggett, H. Medina and K. Merrill develop the notion of "Generalized Multiresolution Analyses" (GMRA's) corresponding to specific M.S.F. wavelets in $L^2(\mathbb{R}^n)$ corresponding to dilation by integer dilation matrix: this comes with a "multiplicity function".

2003–5 – Following on the work of BMM and Bratelli-Jorgensen, Baggett, Jorgensen, Merrill and Packer discuss the notion of "loop group equivalence" for wavelet and frame systems in $L^2(\mathbb{R}^n)$ having the same multiplicity function

2006–8 – Baggett, Larsen, Merrill, Packer, and Raeburn discuss how to construct GMRA's (not necessarily in $L^2(\mathbb{R}^n)$) with a given multiplicity function using direct limits and filter methods

2008–9 – Baggett, Furst, Merrill, Packer compare various constructions of GMRA's, using different notion of equivalence than loop group notion.

Aim of talk: Describe notions of equivalence between GMRA's (based on work with Baggett, Furst, Merrill).

Preliminaries

We first consider dilation, translation operators on $L^2(\mathbb{R})$, corresponding to dilation by integer $N > 1$:

$$\delta(f)(t) = \sqrt{N} f(N(t)),$$

$$T(f)(t) = f(t-1),\ f \in L^2(\mathbb{R}).$$

Note that

$$\delta^{-1} T_v \delta = T_{N(v)},\ \forall v \in \mathbb{Z}.$$

L. Baggett, H. Medina and K. Merrill extended Mallat and Meyer's Multiresolution Analysis, and developed the theory of Generalized Multiresolution Analysis (GMRA) in $L^2(\mathbb{R})$, which generalized the concept of MRA to deal with any wavelet family in $L^2(\mathbb{R}^n)$.

The notion of GMRA can be generalized to abstract Hilbert spaces carrying representations of discrete abelian groups (translations), and "dilation operators". For simplicity in this talk, we keep the translation group $\mathbb{Z}$, and let dilation still be denoted by a unitary operator δ. Let T be a unitary representation of $\mathbb{Z}$ acting in a Hilbert space $\mathcal{H}$, and let δ be a unitary operator on $\mathcal{H}$ for which

$$\delta^{-1}T_v\delta = T_{N(v)}$$

for all $v \in \mathbb{Z}$.

Definition: A collection $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of $\mathcal{H}$ is called a *g*eneralized multiresolution analysis (GMRA) relative to T and δ if

1. $V_j \subseteq V_{j+1}$ for all j.

2. $V_{j+1} = \delta(V_j)$ for all j.

3. $\cap V_j = \{0\}$, and $\cup V_j$ is dense in $\mathcal{H}$.

4. V_0 is invariant under the representation T of $\mathbb{Z}$.

The measure on $\mathbb{T}$ corresponding to T :

We apply the Spectral Theorem for a unitary operator to the subrepresentations of T acting in V_0 and $W_0 = V_1 \cap V_0^{\perp}$.

Consider the restriction of T to V_0. By the Spectral Theorem, up to equivalence of measures, there exists a unique finite Borel measure μ on $\widehat{\mathbb{Z}} = \mathbb{T}$, unique (up to sets of μ measure 0) Borel subsets $\sigma_1 \supseteq \sigma_2 \supseteq \ldots$ of $\mathbb{T}$ and a (not necessarily unique) unitary operator $J : V_0 \to \bigoplus_i L^2(\sigma_i, \mu)$ satisfying

$$[J(T_v(f))](x) = e^{2\pi i v x}[J(f)](x)$$

for all $v \in \mathbb{Z}$, all $f \in V_0$, and μ almost all $x \in \mathbb{T}$.

One similarly comes up with a measure $\widetilde{\mu}$ on subsets of $\mathbb{T}$ associated to the operator T restricted to W_0.

This observation was first made by Baggett, Merrill and Medina for dilations and translations on $L^2(\mathbb{R}^n)$. In this talk, we'll concentrate on the case where the measure μ and $\widetilde{\mu}$ associated to the GMRA are absolutely continous with respect to Haar measure on $\mathbb{T}$, and the multiplicity function m is finite. This is the case in many BMM, BCM examples. Such assumptions are necessary, since L. Baggett has recently constructed examples of GMRA's where μ is atomic.
For the purposes of this talk, we will call such a GMRA a **Haar GMRA**. It has been shown previously that certain function systems satisfying orthogonality relations and other conditions ("low-pass") determine GMRAs.

The multiplicity function for a GMRA

Given a GMRA $\{V_i\}_{i\in\mathbb{Z}}$, the function $m(x) = \sum_j \chi_{\sigma_j}(x)$ defined on $\mathbb{T}$ is called the **multiplicity function** corresponding to the GMRA. It satisfies the inequality

$$m(x) \leq \sum_{l=0}^{N-1} m(\frac{x+l}{N}) \text{ a.e.}$$

Assume m is essentially bounded on $\mathbb{T}$ by $c > 0$. Define the conjugate multiplicity function

$$\tilde{m}(x) = \sum_{l=0}^{N-1} m(\frac{x+l}{N}) - m(x).$$

By definition, m and $\tilde{m}$ satisfy the following **consistency equation**:

$$m(x) + \tilde{m}(x) = \sum_{l=0}^{N-1} m(\frac{x+l}{N}).$$

In the $L^2(\mathbb{R})$ case, Baggett, J. Courter and Merrill constructed scaling functions $\{\phi_1, \phi_2, \cdots, \phi_c\} \subseteq V_0$, defined by the unitary operator $J : V_0 \to \oplus_{i=1}^{c} L^2(\sigma_i)$ as follows:

$$\phi_i = J^{-1}(\chi_{\sigma_i}),\ 1 \leq i \leq c.$$

They also provided an algorithm that shows given a GMRA $\{V_i\}_{i\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ for translation by $\mathbb{Z}$ and dilation by N, with associated multiplicity function m, one can construct a finite tight frame wavelet family $\{\psi_1, \cdots, \psi_d\} \subseteq L^2(\mathbb{R})$. To do this, B C M constructed **generalized low-pass filter functions** $\{h_{i,j}\}_{1\leq i,j \leq c}$, where each $h_{i,j}$ is supported on σ_j, which satisfy the following dilation equations with respect to the generalized scaling functions ϕ_i described earlier, δ, and J :

$$J(\delta^{-1}(\phi_i))(x) = \sum_{j=1}^{c} h_{i,j}\chi_{\sigma_j}(x),\ 1 \leq i \leq c.$$

The χ_{σ_j} are now considered as vectors in the direct sum space $\bigoplus_{i=1}^{c} L^2(\sigma_i)$ which are zero except in the j^{th} component.
Let $\tilde{\sigma_i} = \{x \in \mathbb{T} : \tilde{m}(x) \geq i\}$. Set $d =$ ess sup $\tilde{\mu}(x)$ to get

$$\tilde{\sigma_1} \supseteq \tilde{\sigma_2} \cdots \supseteq \tilde{\sigma_d}.$$

In the $L^2(\mathbb{R})$ set-up, B,C,M also constructed **generalized high-pass filter functions**, $\{g_{k,j}\}_{1\leq k \leq}$ where each $g_{k,j}$ is supported on σ_j.

Key orthogonality conditions
satisfied by generalized filter functions:

The filter functions $\{h_{i,j}\}_{1\leq\ i,j\ \leq c}$ and $\{g_{k,j}\}_{1\leq\ k\ \leq d,\ 1\leq\ j\ \leq c}$, satisfy the following orthogonality relations:

$$\sum_{j=1}^{c}\sum_{l=0}^{N-1} h_{i,j}(\frac{x+l}{N})\overline{h_{i',j}(\frac{x+l}{N})} \ = \ N\delta_{i,i'}\chi_{\sigma_i}(x) \ \ (1),$$

$$\sum_{j=1}^{c}\sum_{l=0}^{N-1} g_{k,j}(\frac{x+l}{N})\overline{g_{k',j}(\frac{x+l}{N})} \ = \ N\delta_{k,k'}\chi_{\widetilde{\sigma_k}}(x) \ \ (2),$$

and

$$\sum_{j=1}^{c}\sum_{l=0}^{N-1} h_{i,j}(\frac{x+l}{N})\overline{g_{k,j}(\frac{x+l}{N})} = 0 \ \ (3)$$

for all i and k.

In the general Haar GMRA setup, the situation is the same: given a Haar GMRA $\{V_i\}_{i\in\mathbb{Z}}$, subsets of a Hilbert space $\mathcal{H}$, with corresponding operators $\{T_v : v \in \mathbb{Z}\}$ and $\delta \in \mathcal{U}(\mathcal{H})$, assuming the multiplicity function m is finite and bounded, one can form filters $H = (h_{i,j})_{1\leq i,j\leq c}$ associated to the behavior of δ^{-1} on V_0, and high-pass filters $G = (g_{k,j})_{1\leq k\leq c,\, 1\leq j\leq d}$, associated to the behavior of δ^{-1} on W_0, satisfying the orthogonality relations (1), (2), (3). Moreover, the filter systems H and G give rise to isometries $S_H : \bigoplus_i L^2(\sigma_i) \to \bigoplus_i L^2(\sigma_i)$ defined by

$$[S_H(f)](x) = H^t(x)f(Nx),$$

$S_G : \bigoplus_k L^2(\widetilde{\sigma}_k, \widetilde{\mu}) \to \bigoplus_i L^2(\sigma_i, \mu)$, defined by

$$[S_G(f)](x) = G^t(x)f(Nx).$$

In work by Baggett, Jorgensen, Merrill, and P., it was shown that the operators S_H and S_G satisfy the following:

1. $S_H^* S_H = I$, $S_G^* S_G = \widetilde{I}$,

2. $S_H^* S_G = 0$, and

3. $S_H S_H^* + S_G S_G^* = I$,

where I is the identity operator on $\bigoplus_i L^2(\sigma_i, \mu)$ and $\widetilde{I}$ is the identity operator on $\bigoplus_k L^2(\widetilde{\sigma}_k, \widetilde{\mu})$.

Using m and H to build a GMRA:

Let $m : \mathbb{T} \to \{0, 1, 2, \ldots\}$ be a Borel function satisfying the consistency inequality $m(x) \leq \sum_{l=0}^{N-1} m(\frac{x+l}{N})$ with respect to Haar measure. We let $\sigma_i = \{x \in \mathbb{T} : \ m(x) \geq i\}$. Recall $\tilde{m}(x) = \sum_{l=0}^{N-1} m(\frac{x+l}{N}) - m(x)$, and $\widetilde{\sigma_k}$ are built from $\tilde{m}$. As before, suppose m is bounded by c, so that $\tilde{m}$ is also bounded, say by d. Let $H = (h_{i,j})$ be a filter relative to m and N, i.e. suppose the functions $h_{i,j}$ satisfy the orthogonality relation (1). Moreover, suppose that the operator $S_H : \bigoplus_i L^2(\sigma_i) \to \bigoplus_i L^2(\sigma_i)$ is a pure isometry.

Conditions on H for this to occur have been described first in by B,C and M, then in B, Jorgensen, M and P, in B, Larsen, M, P and Raeburn, then more recently by B, Furst, M and P. E.g., in BLMPR, H was assumed to be Lipschitz around the identity, and $H(0)$ was assumed to be a diagonal matrix with $\sqrt{N}$'s in the first a elements of the diagonal and 0's elsewhere; this does not always occur in practice (e.g., in MRA's coming from wavelets on fractals). One can obtain far weaker conditions on H guaranteeing S_H is a pure isometry; see BFMP (on ArXiv: 0812.2042).

It is always possible to construct a complementary filter, $G = (g_{k,j})$, relative to m and H, i.e. the functions $G = (g_{k,j})$ will satisfy the orthogonality relations (2) and (3) with respect to H.

Theorem A (BFMP): Given a Borel function $m : \mathbb{T} \to \{0, 1, 2, \ldots\}$ that satisfies the consistency inequality with respect to Haar measure μ on $\mathbb{T}$ and dilation by N, and a filter $H = (h_{i,j})$ satisfying the orthogonality relations (1) for m and N such that the corresponding operator S_H is a pure isometry, and a complementary filter H, it is always possible to construct a GMRA $\{V_j\}$ such that the multiplicity function associated to the unitary operator T on the core subspace V_0 is the given function m, and moreover that the given H is a filter associated to the GMRA $\{V_j\}$.

Idea of construction: The multiplicity functions m and $\tilde{m}$ are used to construct $\{\sigma_i\}$ and $\{\widetilde{\sigma}_j\}$ which are then used to construct the core subspace V_0 and the wavelet/framelet space W_0. The operator T comes from multiplication by complex exponentials. The operators S_H and S_G are used to construct the dilation. As one expects, the intersection 0 property depends in a key way on S_H being a pure isometry. The density condition is forced by the construction. The GMRA here is called the **canonical GMRA** associated to the triple $(m,\ H,\ G)$, and denoted by $\{V_j^{m,H,G}\}$.

Equivalence of GMRAs in terms of the parameters m, H, and G

Let $\{V_j\}$ be a GMRA in a Hilbert space $\mathcal{H}$, relative to a representation T of $\mathbb{Z}$ and a unitary (dilation) operator δ satisfying $\delta^{-1}T_v\delta = T_{N(v)}$ for all $v \in \mathbb{Z}$.

Let $\{V_j'\}$ be a GMRA in a Hilbert space $\mathcal{H}'$, relative to a representation T' of $\mathbb{Z}$ and a unitary operator δ' satisfying the same commutation relations.

Definition: We say that the GMRAs $\{V_j\}$ and $\{V_j'\}$ are **equivalent** if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ that satisfies:

1. $U(V_j) = V_j'$ for all j.

2. $U \circ T_v = T_v' \circ U$ for all $v \in \mathbb{Z}$.

3. $U \circ \delta = \delta' \circ U$.

Lemma B (BFMP):

Let $\{V_j\}$ and $\{V'_j\}$ be as above. Then $\{V_j\}$ and $\{V'_j\}$ are equivalent if and only if there exist operators $P : V_0 \to V'_0$ and $Q : W_0 \to W'_0$ that satisfy:

1. $P \circ T_v = T'_v \circ P$ for all $v \in \mathbb{Z}$.

2. $P \circ \delta^{-1} = \delta'^{-1} \circ P$.

3. $Q \circ T_v = T'_v \circ Q$ for all $v \in \mathbb{Z}$.

4. $\delta'^{-1} \circ Q = P \circ \delta^{-1}$ on W_0.

It turns out that every Haar GMRA is equivalent to one of the canonical GMRAs guaranteed in Theorem A.

Theorem C (BFMP): Let $\{V_j\}$ be a Haar GMRA. Let m be its associated multiplicity function, let H be the filter system relative to m and N, obtained from considering T and δ acting on V_0, and let G be a complementary filter system to H, obtained from considering T and δ on W_0. Then the GMRA $\{V_j\}$ is equivalent to the canonical GMRA $\{V_j^{m,H,G}\}$.

The proof of the Theorem employs the previous Lemma.

We observe the following about a filter system H, which, in the case where the multiplicity function m is bounded by c, is a square $c \times c$ matrix, and even in the case where m is unbounded, can be considered as a square matrix, indexed by $\{i \in \mathbb{N} \cup \{0\} : \ m(x) = i$ on a set of positive measure$\}$. In the study of the GMRA's discussed above, the matrix of filter functions H was used to construct a measurable section from $\widehat{\mathbb{Z}} \cong \mathbb{T}$ into square matrices $H(x)$ of varying dimension depending on $m(x)$.

We remark that the orthogonality relations on filters from H, which can be specialized to

$$\sum_{l=0}^{N-1} \sum_{j} |h_{i,j}(x + \frac{l}{N})|^2 = N\chi_{\sigma_i}(Nx)$$

for almost all $x \in \mathbb{T}$, implies that if $i > m(Nx))$, so that $Nx \notin \sigma_i$, we must have $h_{i,j}(x) = 0$. Likewise, since each $h_{i,j}$ is supported on σ_j, if $j > m(x)$, it will follow that $x \notin \sigma_j$ so that $h_{i,j}(x) = 0$. Thus for any particular value $x \in \mathbb{T}$, the only possible values $\{(i, j)\}$ such that $h_{i,j}(x) \neq 0$ are when $i \leq m(Nx))$ and when $j \leq m(x)$. If we consider only those entries in the matrix $H(x)$, we obtain a $m(Nx)) \times m(x)$ "cut-down" version of the matrix $H(x)$, which we denote by $\widetilde{H}(x)$.

Similarly, we construct a $\widetilde{m}(Nx)) \times m(x)$ "cut-down" version of the matrix $G(x)$, denoted by $\widetilde{G}(x)$.

Theorem C and the above remarks let us reduce the equivalence question for Haar GMRA's to the parameters m, H and G :

<u>**Theorem D (BFMP):**</u> The canonical GMRAs $\{V_j^{m,H,G}\}$ and $\{V'^{m',H',G')}_j\}$ are equivalent if and only if $m = m'$, and there exist matrix-valued functions of varying dimension A and B on $\widehat{\mathbb{Z}^n}$ for which:

1. $A(x)$ is a unitary matrix of dimension $m(x)$.

2. $\widetilde{H}(x)A(x) = A(Nx)\widetilde{H'}(x)$.

3. $B(x)$ is a unitary matrix of dimension $\tilde{m}(x)$.

4. $\widetilde{G}(x)A(x) = B(Nx)\widetilde{G'}(x)$.

For the proof of Theorem D, we use the fact that any unitary operator Λ on a direct sum of vector-valued $L^2(\tau)$ spaces, for $\tau \subset \mathbb{T}$, that commutes with all the multiplication operators $e^{2\pi inx}$, is given as follows:

$$[\Lambda(f)](x) = L(x)\vec{f}(x),$$

where $L(x)$ is a unitary matrix of dimension $m(x) \times m(x)$ acting on the summand $L^2(\tau_r, \mathbb{C}^r)$ for $m(x) = r$.

Loop group equivalence for GMRA's

In 2004–2005, Baggett, Jorgensen, Merrill and P. described a different type of equivalence for GMRA's realized through ordinary translation and dilation on $L^2(\mathbb{R})$.

Let m and $\tilde{m}$ be a multiplicity function and "conjugate" function associated GMRA with related sequences of sets $\{\sigma_i \,:\, 1 \leq\ i\ \leq c\}$ and $\{\tilde{\sigma_k}\ :\ 1\ \leq\ k\ \leq\ d\}$. Suppose $\{h_{i,j}\}_{1\leq\ i,j\ \leq c}$ and $\{g_{k,j}\}_{1\leq\ k\ \leq d,\, 1\leq\ j\ \leq c}$ are generalized low-pass and high-pass filter functions defined above.

Since $\oplus_{j=1}^{c} L^2(\sigma_j) \cong L^2(\bigsqcup_{j=1}^{c} \sigma_j)$, suppress the second index of the filter functions and view generalized filter functions as a vector ($c+d$-tuple) of functions:

$$(h_1, h_2, \cdots h_c, g_1, g_2, \cdots, g_d) \in \oplus_1^{c+d}[L^2(\bigsqcup_{j=1}^{c} \sigma_j)].$$

Following the notation of Bratteli and Jorgensen, we call such a collection an M-**system.**
The orthogonality relations for filters show that we have $h_i(x + \frac{l}{N}) \equiv 0$ as $m(Nx) < i \leq c$ and $g_k(x + \frac{l}{N}) \equiv 0$ as $m(\tilde{N}x) < k \leq d$, for all pairs j, l. Thus for fixed $x \in \mathbb{T}$, combining the first $m(Nx)$ elements of the $\{h_i\}$ and tacking on the first $\tilde{m}(Nx)$ elements of $\{g_k\}$, we have a (column) vector in $[L^2(\bigsqcup_{j=1}^{c} \sigma_j)]^{m(Nx)+\widetilde{m}(Nx)}$.

Definition: Let $p : E \rightarrow \bigsqcup_{j=1}^{c} \sigma_j$ be the vector bundle whose fiber $p^{-1}(x)$ over x is given by the vector space $\mathbb{C}^{m(Nx)+\tilde{m}(Nx)}$.

Any Borel cross-section $M : \bigsqcup_{j=1}^{c} \sigma_j \rightarrow E$ whose coefficient functions, if labeled $M_{i,j}(x) = h_{i,j}(x)$ for $1 \leq i \leq m(Nx)$ and $M_{i,j}(x) = g_{i-m(Nx),j}$ for $m(Nx) + 1 \leq i \leq m(Nx) + \tilde{m}(Nx)$, satisfy the orthogonality relations (1), (2) and (3), which in addition satisfies appropriate "canonical" conditions at the values $\{\frac{l}{N} : 0 \leq l \leq N-1\}$, and is Lipschitz in nhoods of these pts., is called a M-**system**.

In particular, if m is an essentially bounded multiplicity function associated to a GMRA $\{\widehat{V_j'}\} \subset L^2(\mathbb{R})$, with associated generalized low-pass and high-pass filter functions $\{h_{i,j}\}_{1\leq\, i,j\, \leq c}$ and $\{g_{k,j}\}_{1\leq\, k\, \leq d,\, 1\leq\, j\, \leq c}$, the Borel cross section

$$\Big(M_1(x)\,,\ M_2(x)\,,\ \cdots,\ M_{m(Nx)+\tilde{m}(Nx)}(x)\Big)$$

from $\bigsqcup_{i=1}^{c} \sigma_i$ to E defined by $M_i(x) = h_i(x)$ for $1 \leq\ i\ \leq m(Nx)$ and $M_i(x) = g_{i-m(Nx)}$ for $m(Nx) + 1 \leq\ i\ \leq m(Nx) + \tilde{m}(Nx)$, is called the M-system associated to the multiplicity function m and the GMRA $\{\widehat{V_j'}\} \subset L^2(\mathbb{R})$.

All information about the filters $\{h_{i,j}\}$ and $\{g_{k,j}\}$ is encoded in the M-system.

Construct the group bundle $q: F \rightarrow \mathbb{T}$, where the fiber $q^{-1}(x)$ of the bundle consists of the group of complex unitary matrices
$U(m(x) + \tilde{m}(x), \mathbb{C})$. Cross sections to this bundle consist of Borel maps $K : \mathbb{T} \rightarrow F$ such that $q \circ K(x) = x$. We denote the set of sections of this bundle by $\Gamma(F, q)$. Note $\Gamma(F, q)$ is a group under pointwise operations on $\mathbb{T}$. With this, we construct the standard loop group:
$\text{Loop}_m(F, q) =$

$\{U \in \Gamma(F, q) : U(0) = \text{Id}_{m(0)} \text{ in a nhood of } 0\}$.

In 2004, the following was proved about the action of the Loop group on M-systems associated to m;

Theorem (B,J, M, P)

There is a free and transitive action of $\text{Loop}_m(F,q)$ on the set of M-systems associated to an essentially bounded multiplicity function coming from a GMRA. This action is given by
(K, M)

$$\mapsto\ K(Nx)\left[\left(M_1(x), M_2(x), \cdots, M_{m(Nx)+\tilde{m}(Nx)}(x)\right)\right]$$

where $M_i(x) = h_i(x)$ for $1 \leq i \leq m(Nx)$ and $M_i(x) = g_{i-m(x)}$ for $m(Nx)+1 \leq i \leq m(Nx)+\tilde{m}(Nx)$.

Question: Is Loop Group equivalence for filter systems stronger than GMRA equivalence in terms of Theorems C and D, or vice versa?

Answer: Neither of the equivalences implies the other. Loop group equivalence was developed to carry over a variety of low-pass conditions needed to embed GMRA's into the traditional setting of $L^2(\mathbb{R}^n)$. The condition that $U(0) = \text{Id}$ of appropriate dimension is very strong; in addition U needs to be Lip in a nhood of 0. On the other hand, the GMRA equivalence exhibited in Theorem D implies that only H intermingles with H', and only G can intermingle with G', which is stronger that Loop Group equivalence, where the filter system is treated like a column vector and the h's can be combined with the g's.