

WAVELETS IN LITTLEWOOD–PALEY SPACE, AND MEXICAN HAT COMPLETENESS

H.-Q. BUI AND R. S. LAUGESSEN

ABSTRACT. We resolve a long-standing question on completeness of the non-orthogonal Mexican hat wavelet system, in L^p for $1 < p < 2$ and in the Hardy space H^p for $2/3 < p \leq 1$.

Tools include the discrete Calderón condition, a generalization of the Daubechies frame criterion to a weighted L^2 space, and imbeddings of that weighted space into L^p and Hardy spaces.

1. Introduction

Orthogonal wavelets provide universal bases. Y. Meyer discovered that if an orthogonal wavelet possesses some smoothness and decay, then its translates and dilates form an unconditional basis not only for L^2 , but for whole families of Banach spaces including L^p and Hardy spaces. Determining which *non-orthogonal* wavelets are similarly universal seems a difficult task. Even completeness has been unknown for the standard example of the Mexican hat wavelet $\psi(x)$, the second derivative of the negative Gaussian.

Meyer stated in Chapter 4 of his monograph on wavelets and operators [25] that “we do not know whether the functions $2^{j/2}\psi(2^jx - k)$, $j, k \in \mathbb{Z}$, form a complete set in $L^p(\mathbb{R})$ for $1 < p < \infty$.” This Mexican hat spanning problem has been solved only for $p = 2$ by Daubechies [13, p. 75], who proved the Mexican hat system provides a frame for L^2 , which is even stronger than spanning.

For $2 \leq p < \infty$ we recently proved completeness of the Mexican hat system by developing frequency-scale frames in the conjugate space L^q and then imbedding into L^p with the Fourier transform [6].

The remaining case $1 < p < 2$ is resolved in this paper, by extending our frequency-scale theory to a Sobolev space in the frequency domain and then imbedding once more with the Fourier transform. Equivalently, we develop new wavelet expansions in the time domain for a *Littlewood–Paley* (weighted L^2) space, and then imbed that space into L^p and in addition into the Hardy space H^p for $2/3 < p \leq 1$.

Our methods apply not just to the Mexican hat, but to general synthesizers ψ , and provide criteria for “universal” spanning by non-orthogonal wavelet systems. Theorem 1 establishes a bijectivity criterion for the wavelet frame operator on the Littlewood–Paley space. Corollary 2 deduces completeness of wavelet systems in

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L^p and H^p . We verify that criterion for the Mexican hat example at the end of the paper, thus solving Meyer’s problem.

Completeness of the Mexican hat system in L^p for $0 < p < 1$ was proved earlier by a nonlinear approximate identity method of the second author [24, §4.4]. Completeness fails in L^1 , because the Mexican hat and its dilates all have integral zero.

The Mexican hat problem is challenging because the wavelet system is non-orthogonal and non-band limited. Multiresolution analysis does not apply, because the Mexican hat satisfies no scaling or refinement equation.

Wavelet universality results that are relevant to our work include the phi-transform theory of band limited exact dual frames by Frazier and Jawerth [16, 17, 18], the co-orbit theory of Feichtinger and Gröchenig [14, 15, 21], and the approximate duals of Gilbert, Han, Hogan, Lakey, Weiland and Weiss [19]. One can regard our results as relaxing the band limitation of Frazier and Jawerth while avoiding the oversampling inherent in the approaches of Feichtinger and Gröchenig and Gilbert *et al.*

The paper is organized as follows. Analysis and synthesis operators are defined on weighted function and sequence spaces, in Section 2. The frame bijectivity results are developed in Section 3, with proofs in Sections 4–7. The Mexican hat example is treated in Section 8. Open problems are in Section 9.

2. Definitions and assumptions

Define the Fourier transform with 2π in the exponent,

$$\widehat{F}(x) = \int_{\mathbb{R}} F(\xi) e^{-2\pi i \xi x} d\xi, \quad x \in \mathbb{R}.$$

Parseval’s identity says $\langle F, G \rangle_{L^2} = \langle \widehat{F}, \widehat{G} \rangle_{L^2}$, where $\langle F, G \rangle_{L^2} = \int_{\mathbb{R}} F \overline{G} d\xi$.

Fix a *dilation factor* $a \in \mathbb{R}$ with $|a| > 1$ and a *translation step* $b > 0$.

Analysis and synthesis. Take a function $\psi \in L^2(\mathbb{R})$ and rescale it by translation and dilation to obtain

$$\psi_{j,k}(x) = |a|^{j/2} \psi(a^j x - bk), \quad x \in \mathbb{R}.$$

The factor $|a|^{j/2}$ normalizes the rescaling in L^2 .

The *wavelet system* (or *time-scale* or *affine system*) generated by ψ is the collection of functions $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$. Its *synthesis operator* is the map

$$c = \{c_{j,k}\} \mapsto s(c) = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}$$

where the coefficients $c_{j,k}$ are complex numbers.

Given $\phi \in L^2(\mathbb{R})$, the *analysis operator* is the map

$$f \mapsto t(f) = \{b \langle f, \phi_{j,k} \rangle_{L^2}\}_{j,k \in \mathbb{Z}}.$$

(The factor of b is for later convenience.) The analysis operator filters the signal to determine its weighted average values near the lattice points $x = a^{-j}bk$.

Weighted function space. Wavelets are traditionally studied in L^2 , but we will investigate them in the weighted L^2 space

$$K^{1,2}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \|f\|_{K^{1,2}(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + 4\pi^2 x^2) |f(x)|^2 dx < \infty \right\},$$

which is the Fourier image of the Sobolev space $W^{1,2}(\mathbb{R})$ of square integrable functions having one derivative in L^2 . The notation $K^{1,2}$ comes from the literature on Littlewood–Paley spaces [23].

The Littlewood–Paley space imbeds into L^p for $2/3 < p \leq 2$, because

$$\|f\|_{L^p(\mathbb{R})} \leq \|1/\sqrt{1 + 4\pi^2 x^2}\|_{L^{2p/(2-p)}(\mathbb{R})} \|\sqrt{1 + 4\pi^2 x^2} f\|_{L^2(\mathbb{R})} = C_p \|f\|_{K^{1,2}(\mathbb{R})} \quad (1)$$

by Hölder's inequality, where $C_p < \infty$ because $2p/(2-p) > 1$. With $p = 1$, the imbedding shows that functions in $K^{1,2}$ are integrable.

We will be especially interested in the closed subspace of functions with integral zero, denoted

$$K_*^{1,2}(\mathbb{R}) = \left\{ f \in K^{1,2}(\mathbb{R}) : \int_{\mathbb{R}} f(x) dx = 0 \right\}.$$

Weighted sequence space. Our wavelet coefficients will belong to the weighted sequence space

$$\ell^2(a) = \left\{ c : \|c\|_{\ell^2(a)}^2 = \sum_{j,k \in \mathbb{Z}} (1 + a^{-2j}(1 + k^2)) |c_{j,k}|^2 < \infty \right\}.$$

The factor of k^2 corresponds to x^2 in the definition of the Littlewood–Paley space.

3. Wavelet frames and spanning sets

We aim to prove that the mixed frame operator st (analysis followed by synthesis) is bijective whenever the analyzer ϕ and synthesizer ψ satisfy a discrete Calderón condition and have suitably controlled overlaps in the frequency domain.

First we introduce some quantities needed in the theorem. Suppose

$$\phi = \widehat{\Phi} \quad \text{and} \quad \psi = \widehat{\Psi}$$

and let

$$\Theta(\xi) = \xi \Phi'(\xi) \quad \text{and} \quad \Gamma(\xi) = \xi \Phi(\xi).$$

Define

$$\begin{aligned} & \Delta(\Phi, \Psi) \\ &= \sum_{l \neq 0} \left\| \sum_{j \in \mathbb{Z}} |\Phi(\xi a^{-j}) \Psi(\xi a^{-j} - lb^{-1})| \right\|_{L^\infty(\mathbb{R})}^{1/2} \left\| \sum_{j \in \mathbb{Z}} |\Phi(\xi a^{-j} + lb^{-1}) \Psi(\xi a^{-j})| \right\|_{L^\infty(\mathbb{R})}^{1/2}. \end{aligned} \quad (2)$$

Then let

$$\Delta_*(\Phi, \Psi) = \Delta(\Phi, \Psi) + 2\Delta(\Theta, \Psi) + 2\Delta(\Gamma, \Psi').$$

Use the notation $F(\xi) \lesssim G(\xi)$ to mean F/G is bounded.

Theorem 1 (Frame operator bijectivity). *Assume $\Phi, \Psi \in W^{1,2} \cap W^{1,\infty}(\mathbb{R})$ with $\Phi(0) = \Psi(0) = 0$ and $\Delta_*(\Phi, \Psi) < \infty$, and that their derivatives decay near the origin and infinity according to*

$$|\Phi'(\xi)| \lesssim \begin{cases} |\xi|^\varepsilon, & |\xi| \leq 1, \\ |\xi|^{-\varepsilon-5/2}, & |\xi| \geq 1, \end{cases} \quad \text{and} \quad |\Psi'(\xi)| \lesssim \begin{cases} |\xi|^\varepsilon, & |\xi| \leq 1, \\ |\xi|^{-\varepsilon-3/2}, & |\xi| \geq 1, \end{cases}$$

for some $\varepsilon > 0$. Suppose

$$\sum_{j \in \mathbb{Z}} \overline{\Phi(\xi a^{-j})} \Psi(\xi a^{-j}) = 1 \quad \text{for almost every } \xi \in \mathbb{R}. \quad (3)$$

Let $\phi = \widehat{\Phi}$ and $\psi = \widehat{\Psi}$. Then $\|st - id\|_{K_*^{1,2}(\mathbb{R}) \rightarrow K_*^{1,2}(\mathbb{R})} \leq \Delta_*(\Phi, \Psi)$.

Hence if $\Delta_*(\Phi, \Psi) < 1$, then the frame operator st is a bijection on $K_*^{1,2}(\mathbb{R})$, and the norm equivalence $\|f\|_{K_*^{1,2}(\mathbb{R})} \asymp \|t(f)\|_{\ell^2(a)}$ holds for all $f \in K_*^{1,2}(\mathbb{R})$.

The synthesis and analysis operators s and t are shown to be bounded in Section 4, and then Theorem 1 is proved in Section 6. See also the remarks below.

We call ϕ an *approximate dual* to the synthesizer ψ when the mixed frame operator st is within distance < 1 of the identity operator [10]. For an *exact dual* the frame operator would need to equal the identity (giving perfect reconstruction), but exact duals need not exist even in L^2 , since wavelet frames can fail to possess a wavelet-structured dual frame [9, §12.1],[22, §8.3].

Remarks on Theorem 1.

Assumption (3) is the *discrete Calderón condition*. (The discreteness refers to the dilation scales $j \in \mathbb{Z}$.) The Calderón condition is central to our approach, for it suggests how to construct an approximate dual analyzer, and thus leads to our solution of the Mexican hat problem in Section 8. For more discussion of Calderón conditions, see our earlier paper [6].

Theorem 1 generalizes the Daubechies criterion [13, §3.3.2] for a wavelet frame in L^2 . The most significant differences are that Theorem 1:

- (i) applies to a weighted L^2 space, corresponding to a Sobolev space in the frequency domain;
- (ii) allows the analyzer and synthesizer to differ (so that given the synthesizer, we have the freedom to choose a good analyzer); and
- (iii) requires the Calderón expression $\sum_{j \in \mathbb{Z}} \overline{\Phi(\xi a^{-j})} \Psi(\xi a^{-j})$ to equal 1 everywhere.

Daubechies works in L^2 , assumes the analyzer and synthesizer are the same ($\Phi = \Psi$), and requires the Calderón expression only to be bounded away from zero and infinity.

Corollary 2 (Spanning L^p and H^p). *Assume Φ and Ψ satisfy the hypotheses of Theorem 1, with $\Delta_*(\Phi, \Psi) < 1$. Let $\psi = \widehat{\Psi}$.*

Then the wavelet system $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ spans $L^p(\mathbb{R})$ for $1 < p \leq 2$, and spans the Hardy space $H^p(\mathbb{R})$ for $2/3 < p \leq 1$.

Spanning means that the finite linear combinations of the $\psi_{j,k}$ form a dense set.

The corollary is proved in Section 7, by using Theorem 1 and known imbeddings of the Littlewood–Paley space. The corollary is then applied to the Mexican hat problem, in Section 8, by constructing a good analyzer to pair with the Mexican hat synthesizer.

In Corollary 2 the synthesizer ψ has integral zero, because $\Psi(0) = 0$ by hypothesis. Spanning results for synthesizers having *nonzero* integral can be found in [1, 2, 3, 4, 5] for Lebesgue, Hardy and Sobolev spaces.

Related literature. Gilbert *et al.* [19, p. 5] prove the existence of numbers $a > 1$ and $b > 0$ such that the wavelet system $\{\psi(a^j x - bk) : j, k \in \mathbb{Z}\}$ yields a frame operator st that is invertible on $H^1(\mathbb{R})$ and $L^p(\mathbb{R})$, $1 < p < \infty$. Bui and Paluszynski [8, Theorem 3.3] prove the same result for $a = 2$ provided $b > 0$ is sufficiently small. They also treat H^p for $1/2 < p < 1$. Neither paper indicates for which parameter values their theorems apply. In contrast, the dilation and translation parameters in this paper are given, and we do not need to oversample them.

4. Analysis and Synthesis

To prove boundedness of the synthesis and analysis operators s and t , we first recall corresponding results for L^2 .

Proposition 3 (Bounded synthesis into L^2 ; e.g. [6, Prop. 6] or [11, Theorem 2]). *Assume $\Psi \in L^2 \cap L^\infty(\mathbb{R})$ decays near the origin and infinity according to*

$$|\Psi(\xi)| \lesssim \begin{cases} |\xi|^\varepsilon, & |\xi| \leq 1, \\ |\xi|^{-\varepsilon-1/2}, & |\xi| \geq 1, \end{cases}$$

for some $\varepsilon > 0$. Let $\psi = \widehat{\Psi}$.

Then $s : L^2(\mathbb{Z} \times \mathbb{Z}) \rightarrow L^2(\mathbb{R})$ is bounded and linear, with unconditional convergence of the series $s(c) = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}$.

Proposition 4 (Bounded analysis on L^2 ; e.g. [6, Proposition 7]). *Assume $\Phi \in L^2 \cap L^\infty(\mathbb{R})$ decays near the origin and infinity according to*

$$|\Phi(\xi)| \lesssim \begin{cases} |\xi|^\varepsilon, & |\xi| \leq 1, \\ |\xi|^{-\varepsilon-1/2}, & |\xi| \geq 1, \end{cases}$$

for some $\varepsilon > 0$. Let $\phi = \widehat{\Phi}$.

Then $t : L^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{Z} \times \mathbb{Z})$ is bounded and linear.

Now we prove analogous results with weights, for the Littlewood–Paley space.

Proposition 5 (Synthesis into $K_*^{1,2}$). *Assume $\Psi \in W^{1,2} \cap W^{1,\infty}(\mathbb{R})$ with $\Psi(0) = 0$ and that its derivative decays near the origin and infinity according to*

$$|\Psi'(\xi)| \lesssim \begin{cases} |\xi|^\varepsilon, & |\xi| \leq 1, \\ |\xi|^{-\varepsilon-3/2}, & |\xi| \geq 1, \end{cases}$$

for some $\varepsilon > 0$. Let $\psi = \widehat{\Psi}$.

Then $s : \ell^2(a) \rightarrow K_*^{1,2}(\mathbb{R})$ is bounded and linear, with unconditional convergence of the series $s(c) = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}$.

Proof of Proposition 5. Let $c \in \ell^2(a)$. The assumptions on Ψ' imply that Ψ decays like $|\xi|^{\varepsilon+1}$ near the origin and like $|\xi|^{-\varepsilon-1/2}$ near infinity. Hence by Proposition 3, the series $s(c)$ converges unconditionally in $L^2(\mathbb{R})$ with

$$\|s(c)\|_{L^2(\mathbb{R})} \leq (\text{const.}) \|c\|_{\ell^2(\mathbb{Z} \times \mathbb{Z})}.$$

To bound the L^2 norm of $x \cdot s(c)(x)$, we have

$$\begin{aligned} x \cdot s(c)(x) &= x \sum_{j,k \in \mathbb{Z}} c_{j,k} |a|^{j/2} \psi(a^j x - bk) \\ &= \sum_{j,k \in \mathbb{Z}} (a^{-j} c_{j,k}) |a|^{j/2} (a^j x - bk) \psi(a^j x - bk) \end{aligned} \quad (4)$$

$$+ \sum_{j,k \in \mathbb{Z}} (a^{-j} bk c_{j,k}) |a|^{j/2} \psi(a^j x - bk) \quad (5)$$

by substituting $x = a^{-j}(a^j x - bk) + a^{-j}bk$. The first quantity (4) has L^2 norm bounded by $(\text{const.}) \|\{a^{-j} c_{j,k}\}\|_{\ell^2(\mathbb{Z} \times \mathbb{Z})}$, by Proposition 3 applied with $x\psi(x)$ instead of $\psi(x)$; observe that $x\psi(x) = \widehat{\Psi}'(x)/2\pi i \in L^2(\mathbb{R})$ and that Ψ' satisfies the hypotheses of Proposition 3. The second quantity (5) has L^2 norm bounded by $(\text{const.}) \|\{a^{-j} k c_{j,k}\}\|_{\ell^2(\mathbb{Z} \times \mathbb{Z})}$, using Proposition 3 once more.

Combining these estimates, we see $x \cdot s(c)(x)$ has L^2 norm bounded by a constant times $\|c\|_{\ell^2(a)}$. Thus s maps boundedly into $K_*^{1,2}(\mathbb{R})$.

Finally, $\int_{\mathbb{R}} s(c)(x) dx = 0$ because $\int_{\mathbb{R}} \psi(x) dx = \Psi(0) = 0$ and the series for $s(c)$ converges in L^1 , by the imbedding (1). So $s(c) \in K_*^{1,2}(\mathbb{R})$. \square

Proposition 6 (Analysis on $K_*^{1,2}$). *Assume $\Phi \in W^{1,2} \cap W^{1,\infty}(\mathbb{R})$ with $\Phi(0) = 0$ and that its derivative decays near the origin and infinity according to*

$$|\Phi'(\xi)| \lesssim \begin{cases} |\xi|^\varepsilon, & |\xi| \leq 1, \\ |\xi|^{-\varepsilon-5/2}, & |\xi| \geq 1, \end{cases}$$

for some $\varepsilon > 0$. Let $\phi = \widehat{\Phi}$.

Then $t : K_*^{1,2}(\mathbb{R}) \rightarrow \ell^2(a)$ is bounded and linear.

Proof of Proposition 6. We will show that for $f \in K_*^{1,2}(\mathbb{R})$,

$$\sum_{j,k \in \mathbb{Z}} |\langle f, \phi_{j,k} \rangle_{L^2}|^2 \leq (\text{const.}) \|f\|_{L^2(\mathbb{R})}^2, \quad (6)$$

$$\sum_{j,k \in \mathbb{Z}} a^{-2j} |\langle f, \phi_{j,k} \rangle_{L^2}|^2 \leq (\text{const.}) \|xf\|_{L^2(\mathbb{R})}^2, \quad (7)$$

$$\sum_{j,k \in \mathbb{Z}} a^{-2j} k^2 |\langle f, \phi_{j,k} \rangle_{L^2}|^2 \leq (\text{const.}) \|xf\|_{L^2(\mathbb{R})}^2, \quad (8)$$

so that $\|t(f)\|_{\ell^2(a)} \leq (\text{const.})\|f\|_{K_*^{1,2}(\mathbb{R})}$ as desired.

The hypotheses in the proposition imply that Φ decays like $|\xi|^{\varepsilon+1}$ near the origin and like $|\xi|^{-\varepsilon-3/2}$ near infinity. Hence estimate (6) follows from Proposition 4.

To verify estimates (7) and (8), we will express f as a derivative, using that it has integral zero. Let $f \in K_*^{1,2}(\mathbb{R})$ and put $F = \check{f} \in W^{1,2}(\mathbb{R})$. Note $F(0) = \int_{\mathbb{R}} f(x) dx = 0$. Let $G(\xi) = F(\xi)/(-2\pi i\xi)$. Then $\|G\|_{L^2(\mathbb{R})} \leq (\text{const.})\|F'\|_{L^2(\mathbb{R})}$ by Hardy's inequality [27, p. 196]. Since G is square integrable we can define $g = \widehat{G} \in L^2(\mathbb{R})$. Observe $f = g'$ weakly, since $F(\xi) = -2\pi i\xi G(\xi)$. Thus f is a derivative.

To establish (7), first substitute $f = g'$ and compute

$$a^{-j}\langle f, \phi_{j,k} \rangle_{L^2} = -\langle g, (\phi')_{j,k} \rangle_{L^2}$$

by integration by parts. Here the weak derivative ϕ' belongs to $L^2(\mathbb{R})$, since $-2\pi i\xi\Phi(\xi) \in L^2(\mathbb{R})$; note $\xi\Phi(\xi)$ decays like $|\xi|^{\varepsilon+2}$ near the origin and like $|\xi|^{-\varepsilon-1/2}$ near infinity. Hence

$$\sum_{j,k \in \mathbb{Z}} a^{-2j} |\langle f, \phi_{j,k} \rangle_{L^2}|^2 \leq (\text{const.}) \|g\|_{L^2(\mathbb{R})}^2 \quad (9)$$

by Proposition 4 applied with ϕ' instead of ϕ , that is, with $-2\pi i\xi\Phi(\xi)$ instead of Φ . Estimate (7) now follows, because

$$\|g\|_{L^2(\mathbb{R})} = \|G\|_{L^2(\mathbb{R})} \leq (\text{const.})\|F'\|_{L^2(\mathbb{R})} = (\text{const.})\|xf\|_{L^2(\mathbb{R})}.$$

To establish (8), we define $\eta(x) = x\phi(x)$ and observe

$$a^{-j}bk\langle f, \phi_{j,k} \rangle_{L^2} = \langle xf, \phi_{j,k} \rangle_{L^2} - a^{-j}\langle f, \eta_{j,k} \rangle_{L^2}, \quad (10)$$

by substituting $bk = a^jx - (a^jx - bk)$. Notice $\eta = \widehat{\Phi}'(x)/2\pi i$, and recall Φ' decays like $|\xi|^\varepsilon$ near the origin and like $|\xi|^{-\varepsilon-5/2}$ near infinity.

The first term $\langle xf, \phi_{j,k} \rangle_{L^2}$ on the right side of (10) determines a sequence in $\ell^2(\mathbb{Z} \times \mathbb{Z})$, by Proposition 4, with norm bounded by a constant times $\|xf\|_{L^2(\mathbb{R})}$. The second term $a^{-j}\langle f, \eta_{j,k} \rangle_{L^2}$ also determines an ℓ^2 sequence with norm bounded by a constant times $\|xf\|_{L^2(\mathbb{R})}$, by arguing like for (9) above except using η instead of ϕ . Hence (8) is proved. \square

5. Remainder bounds

For use in the next section, we develop bounds on a remainder term in the frequency domain, defined by

$$R(F, \Phi, \Psi)(\xi) = \sum_{j \in \mathbb{Z}} \sum_{l \neq 0} F(\xi + lb^{-1}a^j) \overline{\Phi(\xi a^{-j} + lb^{-1})} \Psi(\xi a^{-j}). \quad (11)$$

Lemma 7 (L^2 remainder estimate). *Let $F \in L^2(\mathbb{R})$ and assume Φ and Ψ are measurable functions with $\Delta(\Phi, \Psi) < \infty$.*

Then $\|R(F, \Phi, \Psi)\|_{L^2(\mathbb{R})} \leq \|F\|_{L^2(\mathbb{R})} \Delta(\Phi, \Psi)$. The series defining $R(F, \Phi, \Psi)$ converges pointwise absolutely almost everywhere, and hence converges unconditionally in $L^2(\mathbb{R})$.

Proof of Lemma 7. See the proof of [6, Theorem 1], with $p = q = 2$. \square

Lemma 7 essentially reproves the remainder estimates in the Daubechies wavelet frame criterion ([13, §3.3.2] and [20, formula (10)]), except that the analyzer and synthesizer here can differ.

Lemma 8 (Sobolev remainder estimate). *Let $F \in W^{1,2}(\mathbb{R})$ with $F(0) = 0$, and assume $\Phi, \Psi \in W^{1,\infty}(\mathbb{R})$ with $\Delta_*(\Phi, \Psi) < \infty$.*

Then $\|R(F, \Phi, \Psi)\|_{W^{1,2}(\mathbb{R})} \leq \|F\|_{W^{1,2}(\mathbb{R})} \Delta_(\Phi, \Psi)$.*

Proof of Lemma 8. First note $R(F, \Phi, \Psi)$ is square integrable by Lemma 7.

Formally differentiating term-by-term in the definition (11) of $R(F, \Phi, \Psi)$, we obtain

$$\begin{aligned} R(F, \Phi, \Psi)'(\xi) &= \sum_{j \in \mathbb{Z}} \sum_{l \neq 0} F'(\xi + lb^{-1}a^j) \overline{\Phi(\xi a^{-j} + lb^{-1})} \Psi(\xi a^{-j}) \\ &\quad + \sum_{j \in \mathbb{Z}} \sum_{l \neq 0} F(\xi + lb^{-1}a^j) \overline{a^{-j} \Phi'(\xi a^{-j} + lb^{-1})} \Psi(\xi a^{-j}) \\ &\quad + \sum_{j \in \mathbb{Z}} \sum_{l \neq 0} F(\xi + lb^{-1}a^j) \overline{a^{-j} \Phi(\xi a^{-j} + lb^{-1})} \Psi'(\xi a^{-j}) \end{aligned} \quad (12)$$

$$= R(F', \Phi, \Psi) + R(G, \Theta, \Psi) + R(G, \Gamma, \Psi') \quad (13)$$

where

$$G(\xi) = F(\xi)/\xi, \quad \Theta(\xi) = \xi \Phi'(\xi), \quad \Gamma(\xi) = \xi \Phi(\xi).$$

Observe $G \in L^2(\mathbb{R})$ with $\|G\|_{L^2(\mathbb{R})} \leq 2\|F'\|_{L^2(\mathbb{R})}$ by Hardy's inequality [27, p. 196], using that $F(0) = 0$.

The first term in (13) satisfies

$$\|R(F', \Phi, \Psi)\|_{L^2(\mathbb{R})} \leq \|F'\|_{L^2(\mathbb{R})} \Delta(\Phi, \Psi)$$

by Lemma 7. Similarly

$$\begin{aligned} \|R(G, \Theta, \Psi)\|_{L^2(\mathbb{R})} &\leq \|G\|_{L^2(\mathbb{R})} \Delta(\Theta, \Psi) \\ &\leq 2\|F'\|_{L^2(\mathbb{R})} \Delta(\Theta, \Psi), \end{aligned}$$

and

$$\begin{aligned} \|R(G, \Gamma, \Psi')\|_{L^2(\mathbb{R})} &\leq \|G\|_{L^2(\mathbb{R})} \Delta(\Gamma, \Psi') \\ &\leq 2\|F'\|_{L^2(\mathbb{R})} \Delta(\Gamma, \Psi'). \end{aligned}$$

From these bounds and Lemma 7, we conclude that the three series in (12) converge unconditionally in L^2 . Consequently (13) holds rigorously in the sense of weak derivatives. Summing our three estimates gives $\|R(F, \Phi, \Psi)'\|_{L^2(\mathbb{R})} \leq \|F'\|_{L^2(\mathbb{R})} \Delta_*(\Phi, \Psi)$. Combining this inequality with the L^2 bound in Lemma 7 completes the proof of the lemma. \square

6. Proof of Theorem 1: bijectivity of the frame operator $K_*^{1,2}(\mathbb{R}) \rightarrow K_*^{1,2}(\mathbb{R})$

The synthesis and analysis operators s and t in Theorem 1 are bounded on the Hilbert spaces $\ell^2(a)$ and $K_*^{1,2}(\mathbb{R})$ respectively, by Propositions 5 and 6. We will show

$$\|st(f) - f\|_{K_*^{1,2}(\mathbb{R})} \leq \Delta_*(\Phi, \Psi) \|f\|_{K_*^{1,2}(\mathbb{R})}, \quad f \in K_*^{1,2}(\mathbb{R}), \quad (14)$$

so that $\|st - \text{id}\| \leq \Delta_*(\Phi, \Psi)$.

Lifting this desired inequality to the frequency domain by Plancherel, it says

$$\|(st(\widehat{F}))^\sim - F\|_{W^{1,2}(\mathbb{R})} \leq \Delta_*(\Phi, \Psi) \|F\|_{W^{1,2}(\mathbb{R})} \quad (15)$$

for $F \in W^{1,2}(\mathbb{R})$ with $F(0) = 0$. A known calculation (e.g. [6, formula (16)]) reveals the lifting of the mixed frame operator to be

$$\begin{aligned} (st(\widehat{F}))^\sim(\xi) &= \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} F(\xi + lb^{-1}a^j) \overline{\Phi(\xi a^{-j} + lb^{-1})} \Psi(\xi a^{-j}) \\ &= F(\xi) \sum_{j \in \mathbb{Z}} \overline{\Phi(\xi a^{-j})} \Psi(\xi a^{-j}) + R(F, \Phi, \Psi)(\xi), \end{aligned}$$

by splitting off the terms with $l = 0$.

The discrete Calderón condition (3) says $\sum_{j \in \mathbb{Z}} \overline{\Phi(\xi a^{-j})} \Psi(\xi a^{-j}) = 1$ a.e., and so $(st(\widehat{F}))^\sim - F = R(F, \Phi, \Psi)$. Now the goal (15) follows from the Sobolev remainder estimate in Lemma 8.

The rest of Theorem 1 follows from (14) and the standard lemma below.

Lemma 9. *Suppose $s : X \rightarrow Y$ and $t : Y \rightarrow X$ are bounded linear operators on the Banach spaces X and Y , with $\|st - \text{id}\| < 1$.*

Then $st : Y \rightarrow Y$ is a bijection, and $\|f\|_Y \asymp \|t(f)\|_X$ for all $f \in Y$.

Proof of Lemma 9. The invertibility of $st = \text{id} - (\text{id} - st)$ is immediate by a Neumann series, and the norm equivalence $\|f\|_Y \asymp \|t(f)\|_X$ holds because

$$\|f\|_Y \leq \|(st)^{-1}s\| \|t(f)\|_X \leq \|(st)^{-1}s\| \|t\| \|f\|_Y.$$

□

7. Proof of Corollary 2, by imbedding Littlewood–Paley into Hardy

The finite linear combinations of the $\psi_{j,k}$ are dense in $K_*^{1,2}(\mathbb{R})$, by the surjectivity of synthesis proved in Theorem 1, and $K_*^{1,2}(\mathbb{R})$ imbeds densely into $L^p(\mathbb{R})$ when $1 < p \leq 2$ and $H^p(\mathbb{R})$ when $2/3 < p \leq 1$, by the next proposition. Hence the $\psi_{j,k}$ span L^p and H^p as desired.

Proposition 10. *The Littlewood–Paley space $K_*^{1,2}(\mathbb{R})$ imbeds densely into the Hardy space $H^p(\mathbb{R})$ for $2/3 < p \leq 1$, and imbeds densely into the Lebesgue space $L^p(\mathbb{R})$ for $1 < p \leq 2$.*

The L^p imbedding was proved already in (1). The density of that imbedding is trivial, since smooth functions with compact support and integral zero belong to the Littlewood–Paley space and are dense in L^p (when $p > 1$).

The imbedding into H^p was proved when $p = 1$ by Coifman and Weiss [12, Theorem C] and when $2/3 < p \leq 1$ by Taibleson and Weiss [28, Theorem 2.9] (taking $q = 2, s = 0$ there), using the molecular theory of Hardy spaces. We provide more accessible proofs below, using maximal functions and Hilbert transforms. The proof by maximal functions is the most appealing to us, as it shows the maximal operator is bounded on the Littlewood–Paley space $K_*^{1,2}$. The proof by Hilbert transforms is the most concise, and shows the Hilbert transform is an isometry on $K_*^{1,2}$.

For density of the imbedding into Hardy space, recall that Schwartz functions with integral zero are dense in $H^p(\mathbb{R})$ (see [26, Chapter III, §5.2(a)]).

Assume $2/3 < p \leq 1$, in the following proofs.

Imbedding into Hardy space by maximal functions.

Let λ be a Schwartz function with nonzero integral, write $\lambda_t(x) = \lambda(x/t)/t$, and define the maximal function $(M_\lambda f)(x) = \sup_{t>0} |(\lambda_t * f)(x)|$. Recall the L^2 maximal inequality

$$\|M_\lambda f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}, \quad f \in L^2(\mathbb{R}), \quad (16)$$

which holds whenever λ is bounded by $C/(1 + |x|)^{1+\epsilon}$ [26, Chapter II, §2.1]. We will extend it to the Littlewood–Paley space, proving

$$\|M_\lambda f\|_{K_*^{1,2}(\mathbb{R})} \leq C\|f\|_{K_*^{1,2}(\mathbb{R})}, \quad f \in K_*^{1,2}(\mathbb{R}). \quad (17)$$

Then using the characterization of Hardy spaces in terms of the maximal function [26, Chapter III, §1.2], we will have the desired imbedding:

$$\begin{aligned} \|f\|_{H^p(\mathbb{R})} &\stackrel{\text{def}}{=} \|M_\lambda f\|_{L^p(\mathbb{R})} \\ &\leq C_p \|M_\lambda f\|_{K_*^{1,2}(\mathbb{R})} \quad \text{by imbedding (1)} \\ &\leq C_p \|f\|_{K_*^{1,2}(\mathbb{R})} \end{aligned}$$

by the maximal inequality (17).

The L^2 part of (17) is handled already by (16), and so it suffices to show

$$\|x(M_\lambda f)(x)\|_{L^2(\mathbb{R})} \leq C\|xf(x)\|_{L^2(\mathbb{R})}, \quad f \in K_*^{1,2}(\mathbb{R}). \quad (18)$$

This inequality does *not* follow from the standard weighted norm inequality for the Hardy–Littlewood maximal function, because the weight x^2 does not lie in the Muckenhoupt class A_2 (see for example [26, Chapter V]). Our proof below succeeds only because of the vanishing moment condition $\int_{\mathbb{R}} f(x) dx = 0$ satisfied by functions in $K_*^{1,2}$.

We start by writing $x = y + (x - y)$, so that

$$\begin{aligned} x(\lambda_t * f)(x) &= \int_{\mathbb{R}} \lambda_t(x - y) y f(y) dy + \int_{\mathbb{R}} \lambda_t(x - y) (x - y) f(y) dy \\ &= (\lambda_t * g)(x) + \int_{\mathbb{R}} t\mu_t(x - y) f(y) dy \end{aligned} \quad (19)$$

where $g(x) = xf(x)$ and $\mu(x) = x\lambda(x)$. The first term $\lambda_t * g$ is bounded pointwise by $M_\lambda g$. For the second term, since the integral of f equals zero we have

$$\begin{aligned} \int_{\mathbb{R}} t\mu_t(x-y)f(y) dy &= \int_{\mathbb{R}} t[\mu_t(x-y) - \mu_t(x)]f(y) dy \\ &= - \int_{\mathbb{R}} \int_0^1 (\mu')_t(x-sy) ds g(y) dy \end{aligned}$$

by the fundamental theorem. Taking the supremum over $t > 0$, we deduce

$$\begin{aligned} \sup_{t>0} \left| \int_{\mathbb{R}} t\mu_t(x-y)f(y) dy \right| &\leq \int_0^1 \sup_{t>0} \int_{\mathbb{R}} \nu_t(x-sy)|g(y)| dy ds \quad \text{where } \nu = |\mu'| \\ &= \int_0^1 s^{-1} \sup_{t>0} \int_{\mathbb{R}} \nu_t(s^{-1}x-y)|g(y)| dy ds \quad \text{by } t \mapsto st \\ &= \int_0^1 s^{-1} (M_\nu |g|)(s^{-1}x) ds. \end{aligned}$$

Combining our estimates on the two terms in (19) now gives

$$\begin{aligned} \|x(M_\lambda f)(x)\|_{L^2(\mathbb{R})} &\leq \|M_\lambda g\|_{L^2(\mathbb{R})} + \int_0^1 s^{-1} \|(M_\nu |g|)(s^{-1}x)\|_{L^2(\mathbb{R})} ds \\ &= \|M_\lambda g\|_{L^2(\mathbb{R})} + 2\|M_\nu |g|\|_{L^2(\mathbb{R})} \\ &\leq C\|g\|_{L^2(\mathbb{R})} = C\|xf(x)\|_{L^2(\mathbb{R})} \end{aligned}$$

by the L^2 maximal inequality. Thus we have proved (18). \square

Imbedding into Hardy space by Hilbert transforms.

We show the Hilbert transform is an isometry on $K_*^{1,2}$, meaning

$$\|Hf\|_{K_*^{1,2}(\mathbb{R})} = \|f\|_{K_*^{1,2}(\mathbb{R})}, \quad f \in K_*^{1,2}(\mathbb{R}).$$

First, $\check{f} \in W^{1,2}(\mathbb{R})$ with $\check{f}(0) = 0$. Notice $-i \operatorname{sign}(\xi)\check{f}(\xi)$ belongs to $W^{1,2}(\mathbb{R})$ and has the same $W^{1,2}$ -norm as \check{f} ; here it is crucial that $\check{f}(0) = 0$, so that multiplying by $\operatorname{sign}(\xi)$ does not introduce a discontinuity. Taking the Fourier transform now shows $Hf \in K_*^{1,2}(\mathbb{R})$, with Hf having the same $K^{1,2}$ -norm as f .

Then using the characterization of Hardy spaces in terms of the Hilbert transform [26, Chapter III, §4.2, §4.3], we have

$$\begin{aligned} \|f\|_{H^p(\mathbb{R})} &\stackrel{\text{def}}{=} \|f\|_{L^p(\mathbb{R})} + \|Hf\|_{L^p(\mathbb{R})} \\ &\leq C_p (\|f\|_{K^{1,2}(\mathbb{R})} + \|Hf\|_{K^{1,2}(\mathbb{R})}) \quad \text{by imbedding (1)} \\ &= 2C_p \|f\|_{K_*^{1,2}(\mathbb{R})} \end{aligned}$$

by the isometry property, which proves the desired imbedding of $K_*^{1,2}$ into H^p . \square

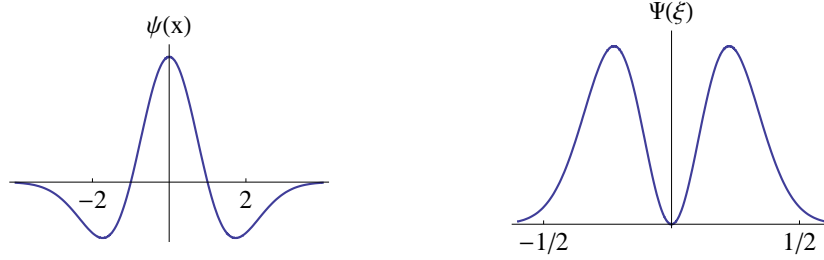


FIGURE 1. The Mexican hat $\psi(x) = -(e^{-x^2/2})'' = (1-x^2)e^{-x^2/2}$, and its Fourier transform $\Psi(\xi) = (2\pi\xi)^2 e^{-(2\pi)^2\xi^2/2}$.

8. The Mexican hat example

Meyer's Mexican hat question in the Introduction concerns the synthesizer $\psi(x) = (1-x^2)e^{-x^2/2}$, which is shown in Figure 1 along with its inverse Fourier transform $\Psi(\xi) = (2\pi\xi)^2 \exp(-2\pi^2\xi^2)$. We work here with dyadic dilations and unit translations, so that

$$a = 2, \quad b = 1.$$

In order to apply our results, we must construct an analyzer for which the discrete Calderón condition (3) holds, that is, $\sum_{j \in \mathbb{Z}} \overline{\Phi(\xi 2^{-j})} \Psi(\xi 2^{-j}) = 1$. We choose Φ to be the “band-limited reciprocal” of Ψ defined by

$$\Phi = \kappa / \Psi$$

where κ is the “double bump” function

$$\kappa(\xi) = \begin{cases} 0, & \xi \in [0, 1/12], \\ \sin^2((12\xi - 1)\pi/2), & \xi \in [1/12, 1/6], \\ \cos^2((6\xi - 1)\pi/2), & \xi \in [1/6, 1/3], \\ 0, & \xi \in [1/3, \infty), \\ \kappa(-\xi), & \xi \in (-\infty, 0). \end{cases}$$

Clearly κ generates a dyadic partition of unity, with $\sum_{j \in \mathbb{Z}} \kappa(\xi 2^{-j}) = 1$ for all $\xi \neq 0$, and so Φ and Ψ satisfy the discrete Calderón condition.

Obviously Φ and Ψ also satisfy the decay assumptions in Theorem 1, with $\Phi(0) = \Psi(0) = 0$.

It is straightforward to estimate by computer that $\Delta_*(\Phi, \Psi) < 0.03$. The bound $\Delta_*(\Phi, \Psi) < 0.52$ can be proved rigorously, if desired [7].

Hence from Theorem 1 and Corollary 2 we conclude that

the Mexican hat system $\{\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ spans $L^p(\mathbb{R})$ for all $1 < p \leq 2$, and the Hardy space $H^p(\mathbb{R})$ for all $2/3 < p \leq 1$.

Thus the Mexican hat completeness problem is solved for $2/3 < p \leq 2$. Recall it was solved for $p = 2$ by Daubechies [13, p. 75], and for $2 < p < \infty$ by the authors [6].

9. Future directions

Is the Mexican hat system complete in H^p for $1/2 < p \leq 2/3$? And when $p \leq 1/2$, can one span using *derivatives* of the Mexican hat, which have additional vanishing moments? Note Bui and Paluszyński [8] answered both questions positively provided the translations are sufficiently oversampled.

Is the mixed frame operator $st(f) = b \sum_{j,k} \langle f, \phi_{j,k} \rangle \psi_{j,k}$ a bijection on L^p for $1 < p < \infty$, under hypotheses that include the Mexican hat synthesizer? If so, then the Mexican hat system can represent each function in L^p by a norm convergent series, which would be stronger than our spanning result in Corollary 2. Note that Tao [29] has shown the frame operator need not be bijective on L^p for $1 < p < 2$, when $\phi = \psi$, even if one supposes the $\psi_{j,k}$ form a Riesz basis and frame for L^2 . Thus some kind of frequency concentration hypothesis will be needed, like we needed $\Delta_*(\Phi, \Psi) < 1$ in Theorem 1.

Can our work extend to the Littlewood–Paley space $K_*^{r,2}$ for $r \neq 1$, corresponding to fractional order Sobolev spaces in the frequency domain? If so, then one could handle Shannon type wavelets whose Fourier transforms are discontinuous. In particular, we ask: does the dyadic Shannon system generated by $\widehat{\psi}(\xi) = \mathbb{1}_{(1/2,1]}(|\xi|)$ span $L^p(\mathbb{R})$ when $1 < p < 2$? The easier range $2 \leq p < \infty$ is discussed in [6, §10.5].

Finally, can the Sobolev remainder estimates in Section 5 be extended to higher dimensions, yielding wavelet frames in some higher dimensional Littlewood–Paley space?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CANTERBURY, CHRISTCHURCH 8020,
NEW ZEALAND

E-mail address: Q.Bui@math.canterbury.ac.nz

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, U.S.A.

E-mail address: Laugesen@illinois.edu