INFINITE MATRICES IN SAMPLING THEORY AND RELATED APPLICATIONS

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In these notes I outline some of the results I mentioned in my talk. The proofs will appear in the forthcoming paper with A. Aldroubi and A. Baskakov.

1. From Frames to Matrices

Definition 1.1. Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\varphi_n \in \mathcal{H}$, $n \in \mathbb{Z}^d$, is a *frame* in $\mathcal{H}$ if for some $0 < a \leq b < \infty$

$$a \|f\|^2 \leq \sum_{n\in\mathbb{Z}^d} |\langle f, \varphi_n\rangle|^2 \leq b \|f\|^2 \tag{1.1}$$

for all $f \in \mathcal{H}$.

The operator $T : \mathcal{H} \to \ell^2$, $Tf = \{\langle f, \varphi_n\rangle\}_{n\in\mathbb{Z}^d}$, $f \in \mathcal{H}$, is called an *analysis operator*. It is an easy exercise to show that a sequence $\varphi_n \in \mathcal{H}$ is a frame in $\mathcal{H}$ if and only if its analysis operator has a left inverse. The adjoint of the analysis operator, $T^* : \ell^2 \to \mathcal{H}$, is given by $T^*c = \sum_{n\in\mathbb{Z}^d} c_n\varphi_n$, $c = (c_n) \in \ell^2$. The *frame operator* is $T^*T : \mathcal{H} \to \mathcal{H}$, $T^*Tf = \sum_{n\in\mathbb{Z}^d} \langle f, \varphi_n\rangle\varphi_n$, $f \in \mathcal{H}$. Again, a sequence $\varphi_n \in \mathcal{H}$ is a frame in $\mathcal{H}$ if and only if its frame operator is invertible. The canonical dual frame $\tilde{\varphi}_n \in \mathcal{H}$ is then $\tilde{\varphi}_n = (T^*T)^{-1}\varphi_n$ and the (canonical) *synthesis operator* is $T^\sharp : \ell^2 \to H$, $T^\sharp = (T^*T)^{-1}T^*$, so that

$$f = T^\sharp Tf = \sum_{n\in\mathbb{Z}^d} \langle f, \varphi_n\rangle\tilde{\varphi}_n = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{\varphi}_n\rangle\varphi_n$$

for all $f \in \mathcal{H}$.

In general Banach spaces one cannot use just the equivalence of norms similar to (1.1). The above construction breaks down because, in this case, the analysis operator ends up being bounded below and not necessarily left invertible. As a result a "frame decomposition" remains possible but "frame reconstruction" no longer makes sense. In case of localizaed frames, however, this obstruction does not exist.

Date: January 11, 2007.

Definition 1.2. A sequence $\varphi^n = (\varphi^n_m)_{m\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d, \mathcal{H})$, $n \in \mathbb{Z}^d$, is a *p-frame* (for $\ell^p(\mathbb{Z}^d, \mathcal{H})$) for some $p \in [1, \infty)$ if

$$a\,\|f\|^p \le \sum_{n\in\mathbb{Z}^d} \left| \sum_{m\in\mathbb{Z}^d} \langle f_m, \varphi^n_m\rangle \right|^p \le b\,\|f\|^p \tag{1.2}$$

for some $0 < a \le b < \infty$ and all $f = (f_m)_{m\in\mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d, \mathcal{H})$. If

$$a\,\|f\| \le \sup_{n\in\mathbb{Z}^d} \left| \sum_{m\in\mathbb{Z}^d} \langle f_m, \varphi^n_m\rangle \right| \le b\,\|f\| \tag{1.3}$$

for some $0 < a \le b < \infty$ and all $f = (f_m)_{m\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d, \mathcal{H})$, then the sequence φ^n is called an ∞-frame. It is called a 0-frame if (1.3) holds for all $f \in \mathfrak{c}_0(\mathbb{Z}^d, \mathcal{H})$.

The operator $T_\varphi = T : \ell^p(\mathbb{Z}^d, \mathcal{H}) \to \ell^p(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d, \mathbb{C})$, given by

$$Tf = \langle f, \varphi_n\rangle := \{ \sum_{m\in\mathbb{Z}^d} \langle f_m, \varphi^n_m\rangle \}_{n\in\mathbb{Z}^d}, \quad f \in \ell^p(\mathbb{Z}^d, \mathcal{H}),$$

is called a *p-analysis operator*, $p \in [1, \infty]$. The 0-analysis operator is defined the same way for $f \in \mathfrak{c}_0(\mathbb{Z}^d, \mathcal{H})$. The properties of a p-frame are, obviously, determined by the properties of its analysis operator.

Definition 1.3. A p-frame φ^n with the p-analysis operator T, $p \in \{0\} \cup [1, \infty]$, is (s, α)-*localized* for some $s > 0$ and $\alpha \ne 0$, if there exists an isomorphism $J : \ell^\infty(\mathbb{Z}^d, \mathcal{H}) \to \ell^\infty(\mathbb{Z}^d, \mathcal{H})$ which leaves invariant $\mathfrak{c}_0$ and all $\ell^q(\mathbb{Z}^d, \mathcal{H})$, $q \in [1, \infty)$, and such that

$$TJ_{|\ell^p} \in \Sigma^\omega_\alpha,$$

where $\omega = (1 + |n|)^s$, $n \in \mathbb{Z}^d$, and Σ^ω_α is a class of α-slanted matrices with certain "off-slant" decay, which depends on the weight ω (see the definition in the end of the notes).

Theorem 1.1. *Let φ^n, $n \in \mathbb{Z}^d$, be an (s, α)-localized p-frame for some $p \in \{0\} \cup [1, \infty]$ with $s > d^2 + d$. Then*

(i) *The q-analysis operator T is well defined and left invertible for all $q \in \{0\} \cup [1, \infty]$, and the q-synthesis operator $T^\sharp = (T^*T)^{-1}T^*$ is also well defined for all $q \in \{0\} \cup [1, \infty]$.*
(ii) *The sequence φ^n, $n \in \mathbb{Z}^d$, and its dual sequence $\tilde{\varphi}^n = (T^*T)^{-1}\varphi^n$, $n \in \mathbb{Z}^d$, are both (s, α)-localized q-frames for all $q \in \{0\} \cup [1, \infty]$.*

(iii) *In* $\mathfrak{c}_0$ *and* ℓ^q, $q \in [1,\infty)$, *we have the reconstruction formula*

$$f = T^\sharp T f = \sum_{n \in \mathbb{Z}^d} \langle f, \varphi_n \rangle \tilde{\varphi}_n = \sum_{n \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_n \rangle \varphi_n.$$

For $f \in \ell^\infty$ *the reconstruction formula remains valid provided the convergence is understood in the weak*-topology.*

Theorem 1.1(iii) shows that an (s,α)-localized p-frame is a Banach frame for $\mathfrak{c}_0$ and all ℓ^q, $q \in [1,\infty]$, in the sense of the following definition.

Definition 1.4 (Gröchenig)**.** A countable sequence $\{x_n\}_{x_n \in J} \subset X'$ in the dual of a Banach space X is a Banach frame for X if there exist an associated sequence space $X_d(J)$, a constant $C \geq 1$, and bounded operator $R : X_d \to X$ such that for all $f \in X$

$$\frac{1}{C}\|f\|_X \leq \|\langle f, x_n \rangle\|_{X_d} \leq C\|f\|_X,$$

$$R(\langle f, x_n \rangle_{j \in J}) = f.$$

The slant α may be viewed as a measure of (relative or absolute) redundancy of a p-frame.

In the theory of localized frames introduced by K. Gröchenig it is possible to extend a localized (Hilbert) frame to Banach frames for the associated Banach spaces. The technique we developed allows us to start with a localized p-frame and deduce that it is, in fact, a Banach frame for the associated Banach spaces. Slanted matrices provide us with additional information which makes it possible to shift emphasis from the frame operator T^*T to the analysis operator T itself.

2. From Sampling to Matrices

The sampling/reconstruction problem includes devising efficient methods for representing a signal (function) in terms of a discrete (finite or countable) set of its samples (values) and reconstructing the original signal from its samples.

Here we assume that the signal is a function f that belongs to a shift invariant space

$$V^p(\varphi) = \left\{ \sum_{k \in \mathbb{Z}^d} c_k \varphi_k,\ c = (c_k) \in \ell^p(\mathbb{Z}^d) \right\},$$

for $p \in [1, \infty]$ or

$$V^0(\varphi) = \left\{ \sum_{k \in \mathbb{Z}^d} c_k \varphi_k,\ c = (c_k) \in \mathfrak{c}_0(\mathbb{Z}^d) \right\},$$

if $p = 0$. The function φ is assumed to be such that its translates $\varphi_k = \varphi(\cdot - k)$ form a p-Riesz basis for $V^p(\varphi)$, thus, making it a closed subspace of L^p.

Sampling is assumed to be performed by a countable collection of finite complex Borel measures $\boldsymbol{\mu} = \{\mu_j\}_{j \in \mathbb{Z}^d} \subset \mathcal{M}(\mathbb{R}^d)$, which is referred to as a *sampler*. If $\boldsymbol{\mu}$ is a collection of Dirac measures then it is called an *ideal* sampler. Otherwise, it is an *average* sampler. A $\boldsymbol{\mu}$-*sample* is a sequence $f(\boldsymbol{\mu}) = \int f d\mu_j$, $j \in \mathbb{Z}^d$. It is not hard to see that if $f \in V^p(\Phi)$ and $\boldsymbol{\mu} \in \ell^\infty(\mathbb{Z}^d, \mathcal{M}(\mathbb{R}^d))$ then $f(\boldsymbol{\mu}) \in \ell^p(\mathbb{Z}^d)$. One of the main goals of sampling theory is to determine when a sampler $\boldsymbol{\mu}$ is *stable*, that is when f is uniquely determined by its $\boldsymbol{\mu}$-sample and a small perturbation of the sampler in $\ell^\infty(\mathbb{Z}^d, \mathcal{M}(\mathbb{R}^d))$ results in a small perturbation of $f \in V^p(\Phi)$. An equivalent form of the above condition is in the following definition.

Definition 2.1. A sampler $\boldsymbol{\mu} \in \ell^\infty(\mathbb{Z}^d, \mathcal{M}(\mathbb{R}^d))$ is *stable on* $V^p(\varphi)$ if the *sampling operator* $\varphi_\mu : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$ defined by its matrix $\varphi_k(\boldsymbol{\mu}) = \int \varphi_k d\mu_j$, $j, k \in \mathbb{Z}^d$:

$$(\Phi_\mu c)(j) = \sum_{k \in \mathbb{Z}^d} \int c_k \varphi_k d\mu_j,\ c \in \ell^p(\mathbb{Z}^d)$$

is bounded below in ℓ^p (or p-bb).

We assume also that the sampler $\boldsymbol{\mu}$ is *separated* so that the operator φ_μ is bounded on $\mathfrak{c}_0$ and all ℓ^p, $p \in [1, \infty]$.

Theorem 2.1. *Assume that* $\omega(n) = (1 + |n|)^s$, $n \in \mathbb{Z}^d$, $s > d^2 + d$, *and* $\boldsymbol{\mu} \in \ell^\infty(\mathbb{Z}^d, \mathcal{M}(\mathbb{R}^d))$ *is separated. Assume also that the sampling operator* φ_μ *is* p*-bb for some* $p \in [1, \infty]$ *and* $\varphi_\mu \in \Sigma_\alpha^\omega$ *for some* $\alpha \in (0, 1]$. *Then* $\boldsymbol{\mu}$ *is a stable sampler on* $V^q(\Phi)$ *for every* $q \in \{0\} \cup [1, \infty]$.

In case of ideal sampling the above result means that under certain very natural conditions a stable set of sampling for some p is, in fact, a (stable) set of sampling for *all* $q \in \{0\} \cup [1, \infty]$. Moreover, using the proof of the theorem one can obtain explicit estimates for the sampling lower bound in ℓ^q given the corresponding lower bound in ℓ^p.

3. Boundedness below of slanted matrices.

Both of the theorems above are direct corollaries of the following very general result for bi-infinite matrices with operator entries.

Theorem 3.1. *Let $s > d^2 + d$ and $\omega(n) = (1 + |n|)^s$. Then $\mathbb{A} \in \Sigma_\alpha^\omega$ is p-bb for some $p \in [1, \infty]$ if and only if $\mathbb{A}$ is q-bb for all $q \in [1, \infty]$.*

Below is the definition of the class Σ_α^ω.

For each $n \in \mathbb{Z}^d$ we let X_n and Y_n be (complex) Banach spaces and $\ell^p = \ell^p(\mathbb{Z}^d, (X_n))$ be the Banach space of sequences $x = (x_n)_{n\in\mathbb{Z}^d}$, $x_n \in X_n$, with the norm $\|x\|_p = \left(\sum_{n\in\mathbb{Z}^d} \|x_n\|_{X_n}^p\right)^{\frac{1}{p}}$ when $p \in [1, \infty)$ and $\|x\|_\infty = \sup_{n\in\mathbb{Z}^d} \|x_n\|_{X_n}$. By $\mathfrak{c}_0 = \mathfrak{c}_0(\mathbb{Z}^d, (X_n))$ we denote the subspace of ℓ^∞ of sequences vanishing at infinity, that is $\lim_{|n|\to\infty} \|x_n\| = 0$, where $|n| = \max_{1\le k\le d} |n_k|$, $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$.

Definition 3.1. For $\alpha \neq 0$ and $j \in \mathbb{Z}^d$ the matrix $A_j = A_j^\alpha = (a_{mn}^{(j)})$, $m, n \in \mathbb{Z}^d$, defined by

$$a_{mn}^{(j)} = a_{mn} \prod_{k=1}^{d} \chi_{[j_k, j_k+1)}(\alpha m_k - n_k)$$

is called the *j-th α-slant* of $\mathbb{A}$.

Observe that for every $m \in \mathbb{Z}^d$ there is at most one $n \in \mathbb{Z}^d$ such that $a_{mn}^{(j)} \neq 0$ and at most K different numbers $\ell \in \mathbb{Z}^d$ such that $a_{\ell m}^{(j)} \neq 0$.

Definition 3.2. The class Σ_α^ω of matrices with *ω-summable* α-slants consists of matrices $\mathbb{A}$ such that $\|\mathbb{A}\|_{\Sigma_\alpha^\omega} = K \sum_{j\in\mathbb{Z}^d} \|A_j\|_{\sup} \omega(j) < \infty$, where ω is a weight.

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