Discrepancy, Riesz sequences of exponentials, and the Feichtinger conjecture

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A sequence $\{e_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ is a *frame* if $\exists A, B > 0$

$$A\|h\|^2 \leq \sum_{i \in I} |\langle h, e_i \rangle|^2 \leq B\|h\|^2 \qquad \forall h \in \mathcal{H}.$$

A frame $\{e_i : i \in I\}$ is *bounded* if

$$\inf_{i \in I} \|e_i\| > 0.$$

Note that it is automatic that $\sup_{i \in I} \|e_i\| < \infty$.

$\{e_i : i \in I\} \subset \mathcal{H}$ is a *Riesz basic sequence* if $\exists K_1, K_2 > 0$ such that for every family of scalars $\{a_i : i \in I\}$ with finitely many nonzero terms

$$K_1 \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i e_i \right\|^2 \leq K_2 \sum_{i \in I} |a_i|^2.$$

Note that for Riesz basic sequences

$$0 < \inf_{i \in I} \|e_i\| \leq \sup_{i \in I} \|e_i\| < \infty.$$

Proposition 1. *A finite union of Riesz basic sequences is a bounded frame for its closed linear span.*

Conjecture 2. *(Feichtinger) Every bounded frame can be written as the finite union of Riesz basic sequences.*

Paving Conjecture: For every bounded linear operator $S : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ whose diagonal is 0 and every $\epsilon > 0$, there is a partition $\{\sigma_1, \ldots, \sigma_M\}$ of $\mathbb{Z}$ such that

$$\|P_{\sigma_i} S P_{\sigma_i}\| < \epsilon \quad \forall 1 \leq i \leq M$$

Proposition 3. *A frame $\{e_i : i \in I\}$ can be written as the finite union of Riesz basic sequences if the operator S whose matrix is given by*

$$S_{ij} = \begin{cases} < e_i, e_j > & i \neq j \\ 0 & i = j \end{cases}$$

can be paved, i.e. if for all $\epsilon > 0$ there exists a partition of I $\sigma_1, \ldots, \sigma_N$ such that $\|P_{\sigma_i} S P_{\sigma_i}\| < \epsilon$.

My question: For $U \subset [0,1]$, the set $\{e^{2\pi i n \xi} 1_U : n \in \mathbb{Z}\}$ is a frame for $L^2(U)$. Question: can this set be partitioned into Riesz sequences? (Still open.)

Definition: For $K \subset \mathbb{Z}$ define

$$D(K) = \lim_{n \to \infty} \frac{\#(K \cap [-n,n])}{2n}$$

if it exists.

Theorem 4 (Bourgain-Tzafriri). *Let $U \subset \mathbb{T}$ be a set of positive measure. There exists a set $K \subset \mathbb{Z}$ such that $D(K) > 0$ and $\{e^{2\pi i k x} \chi_U : k \in K\}$ is a Riesz basic sequence.*

Theorem 5. *If U contains an interval of length $\frac{1}{M}$, then $\{e^{2\pi in\xi}1_U : n \in M\mathbb{Z}+p\}$ is a Riesz sequence for all $0 \leq p < M$. In particular, $\{e^{2\pi in\xi}1_U : n \in \mathbb{Z}\}$ can be partitioned into Riesz sequences.*

Theorem 6 (Bourgain-Tzafriri). *Let $\epsilon_n < 8^{-n}$, and $\{r_n\}$ be an enumeration of the rationals in $[0,1]$. Let $\tilde{U} = \cup_{n\in\mathbb{N}} B_{\epsilon_n}(r_n)$. Then $\{e^{2\pi in\xi}1_U : n \in \mathbb{Z}\}$ can be partitioned into Riesz sequences.*

Naive approach: Partition $\mathbb{Z}$ randomly into K_1, K_2 in the following way: for each $n \in \mathbb{Z}$ flip a coin. If heads, put $e^{2\pi i n\xi}$: $n \in \mathbb{Z}\}$ into K_1 and if tails, put it in K_2.

Using discrepancy, we construct a set $U \subset [0,1]$ such that the probability of such a partition being a partition into Riesz sequences is 0.

Discrepancy.

Let $u = \{u_n : n = 1, 2, \ldots\}$ be a sequence of points in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For a finite, nonempty set $S \subset \mathbb{T}$ (possibly with multiplicity), we define the *discrepancy* of S to be

$$\mathrm{Discr}(S) = \sup_{I \subset \mathbb{T}} \left| \frac{\#(S \cap I)}{\#S} - |I| \right|.$$

The N-term discrepancy of the sequence u is defined as

$$D_N(u) = \mathrm{Discr}(\{u_n : 1 \leq n \leq N\}).$$

Theorem 7. *[Koksma's Inequality] For any sequence of points $u_1, \ldots, u_N$ in $\mathbb{T}$, and any function $f : \mathbb{T} \to \mathbb{R}$ of bounded variation,*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(u_n) - \int_0^1 f(t)\, dt \right| \leq \mathrm{Var}(f)\, \mathrm{Discr}(\{u_n : 1 \leq n \leq N\}),$$

where $\mathrm{Var}(f)$ is the total variation of f.

Theorem 8. *For any $\epsilon > 0$, the N-term discrepancy of $u_n = n\alpha \mod 1$ satisfies*

$$D_N(u) \leq C_\alpha N^{-1} \log^{2+\epsilon} N$$

for almost all α, where C_α depends only on α.

Theorem 9 (BS). *There exists a set $U \subset [0,1]$ such that if $\mathbb{Z}$ is randomly partitioned in to K_1, K_2, then with probability 1, $\{e^{2\pi i n\xi} 1_U : n \in K_j\}$ is a Riesz sequence.*

Let $u = u_N(\xi) = \{\xi, 2\xi, \ldots, N\xi\}$. By Theorem 8 with $\epsilon = 1$, for almost all ξ and all N,

$$D_N(u) \leq \frac{C(\xi)}{N} \log^3 N.$$

Choose K such that

$$0 < |\{\xi \in [0,1] : C(\xi) \leq K\}| < 1,$$

and let $U = \{\xi \in [0,1] : C(\xi) \leq K\}$. This U works.

If $\mathbb{Z}$ is randomly partitioned into K_1, K_2, then with probability 1, for each M, there is an N such that $\{N+1, N+2, \ldots, N+M\} \subset K_j\}$.

Let $f(u) = e^{2\pi i u}$. By Theorem 7,

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(k\xi) - 0 \right| \leq \mathsf{Var}(f)\, \mathsf{Discr}(u_N(\xi)).$$

Therefore,

$$\left| \frac{1}{N} e^{2\pi i N\xi} \sum_{k=1}^{M} f(k\xi) \right| \leq 2\pi K \frac{1}{N} \log^3 N.$$

Multiplying both sides by $\sqrt{N}$ and integrating yields

$$\left(\int_U \left| \frac{1}{\sqrt{N}} e^{2\pi i N\xi} \sum_{k=1}^{M} f(k\xi) \right|^2 \right)^{1/2} \leq |U|^{1/2} 2\pi K N^{-1/2} \log^3 N \to 0 \text{ as } N \to \infty.$$

So, since $\sum_{k=1}^{N} \frac{1}{\sqrt{N}}^2 = 1$, it follows from the definition of Riesz basis that $\{f(x+k) : k \in K_j\}$ is not a Riesz basic sequence.