Recitation problems 12/03/09

Note Title

Contractions

Recall If $\vec{X}$ is a vector field on a region $R \subseteq \mathbb{R}^n$ and σ is a k-form defined on R, the **contraction** $\iota(\vec{X})\sigma$ is a k-1 form defined by

$$\left(\iota(\vec{X})\sigma\right)(\vec{v}_1, \dots \vec{v}_{k-1}) = \sigma(\vec{X}, \vec{v}_1, \dots \vec{v}_{k-1})$$

for any vector fields $\vec{v}_1, \dots \vec{v}_k$.

Note If σ is a 1-form $\iota(\vec{X})\sigma = \sigma(\vec{X})$ is a function.

Example $\sigma = x^2 dx + (x+y)dy + (z+y)dz$

$\vec{X} = \sin y\,\vec{i} + \cos y\,\vec{j} + \tan z\,\vec{k}$.

Then $\iota(\vec{X})\sigma = x^2 dx(\vec{X}) + (x+y)\,dy(\vec{X}) + (z+y)\,dz(\vec{X}) =$

$= x^2 \sin y + (x+y)\cos y + (z+y)\tan z$.

Theorem For any 1-forms $\alpha_1, \dots \alpha_k$ and any vector field $\vec{X}$

$\iota(\vec{X})\alpha_1 \wedge \dots \wedge \alpha_k = \alpha_1(\vec{X})\,\alpha_2 \wedge \dots \wedge \alpha_k - \alpha_2(\vec{X})\,\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \dots \wedge \alpha_k +$

$+ \alpha_3(X)\,\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \dots \wedge \alpha_k - \dots$

Exercises Compute the following contractions:

1) $\iota(-y\vec{i} + x\vec{j})\left(\frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy\right)$

2) $\iota(x\vec{i} + y\vec{j})\,dx \wedge dy$

3) $\iota(x^2\vec{i} + y^2\vec{j} + z\vec{k})\,dx \wedge dy \wedge dz$

4) $\iota(P\vec{i} + Q\vec{j} + R\vec{k})\,dx \wedge dy \wedge dz$. Does the answer look familiar?

5) $\iota(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$

For any map $F: \mathbb{R}^n \to \mathbb{R}^n$, $F(u_1 \ldots u_n) = (x_1(u_1 \ldots u_n), x_2(u_1 \ldots u_n), \ldots x_n(u_1 \ldots u_n))$

$F^*(dx_1 \wedge \ldots \wedge dx_n) = f(u_1 \ldots u_n)\, du_1 \wedge \ldots \wedge du_n$

for some function f. [It turns out that $f(u_1, \ldots, u_n) = \frac{\partial(x_1 \ldots x_n)}{\partial(u_1 \ldots u_n)}$]

F is **orientation-preserving** if $f(u_1 \ldots u_n) > 0$ for all $(u_1 \ldots u_n)$. Check to see if the following maps are orientation-preserving:

1. $F(u_1, u_2) = (u_1 + u_2,\ u_1 - u_2)$
2. $F(u_1, u_2) = (u_1 \cos u_2,\ u_1 \sin u_2)$ defined on $R = \{(u_1, u_2) \mid u_1 > 0\}$
3. $F(u_1, u_2) = (e^{u_1} \cos u_2,\ e^{u_1} \sin u_2,\ u_3 + u_1^2)$
4. $F(u_1, u_2, u_3, u_4) = (u_1 - u_2,\ u_2 - u_3,\ u_3 - u_4,\ u_4 - u_1)$

Recall the change of variables formula:

$$(*) \quad \int_{V = F(u)} \!\!\cdots\! \int a(x_1 \ldots x_n)\, dx_1 \ldots dx_n = \int_u \!\cdots\! \int a(x_1(u_1 \ldots u_n), \ldots x_n(u_1 \ldots u_n)) \left| \frac{\partial(x_1 \ldots x_n)}{\partial(u_1 \ldots u_n)} \right| du_1 \ldots du_n$$

If F is orientation-preserving, $(*)$ becomes

$$\int_{F(u)} \!\!\cdots\! \int a(x_1 \ldots x_n)\, dx_1 \wedge \ldots \wedge dx_n = \int_u \!\cdots\! \int F^*(a\, dx_1 \wedge \ldots \wedge dx_n)$$

Or, even more succinctly, $\boxed{\int_{F(u)} \sigma = \int_u F^* \sigma}$ for any n-form σ

1. Compute $\int_R e^{\frac{x-y}{x+y}}\, dx \wedge dy$, where $R = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x \geq 0 \\ y \geq 0 \\ x+y \leq 1 \end{array} \right\}$

2. Compute $\int_R \sin(x+y) \cos(x-2y)\, dx \wedge dy$ where

$R = \{(x, y) \mid x - 3 < 2y < x,\ -x < y < 3 - x\}$.