§7.1

$X(s,t) = (s^2-t^2, s+t, s^2+3t)$

#1a)

normal to surface at $X(2,-1)$ is

$\underline{n} = X_s(2,-1) \times X_t(2,-1)$ :

$X_s = (2s, 1, 2s)$ , $X_s(2,-1) = (4,1,4)$
$X_t = (-2t, 1, 3)$ , $X_t(2,-1) = (-4,1,3)$

$\underline{n} = X_s(2,-1) \times X_t(2,-1) =$ " $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 4 \\ -4 & 1 & 3 \end{vmatrix}$ " $= (-1, -28, 8)$

#4ac) $\underline{X}(s,t) = (s^2\cos t, s^2\sin t, s)$ $-3 \le s \le 3$
$0 \le t \le 2\pi$

$\underline{X}_s = (2s\cos t, 2s\sin t, 1)$ $X_s(-1,0) = (-2, 0, 1)$
~~$X_t$~~
$\underline{X}_t = (-s^2\sin t, s^2\cos t, 0)$ $X_t(-1,0) = (0, 1, 0)$

a) $\underline{n} = \underline{X}_s \times \underline{X}_t = (-1, 0, -2)$

c) $x = s^2\cos t$
$y = s^2\sin t$
$z = s$

$\underbrace{x^2+y^2-z^2}_{F(x,y,z)} = 0$

8) $X(s,t) = (\overbrace{2\sin s \cos t}^{x}, \overbrace{3\sin s \sin t}^{y}, \overbrace{\cos s}^{z})$

$0 \le s \le \pi \qquad 0 \le t \le 2\pi$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = \sin^2 s \cos^2 t + \sin^2 s \sin^2 t + \cos^2 s$$
$$= 1$$

12) $X(s,t) = (s^3, t^3, st)$

a) Find $s,t$ such that $X(s,t) = (1,-1,-1)$:

$s = 1, \ t = -1$

So $\underline{x}_0 = X(1,-1)$ is a point on the plane

and $\underline{n} = X_s(1,-1) \times X_t(1,-1)$ is a normal to

the plane: $X_s = (3s^2, 0, t) \qquad X_s(1,-1) = (3,0,-1)$
$X_t = (0, 3t^2, s) \qquad X_t(1,-1) = (0,3,1)$
$\therefore \ \underline{n} = (3,0,-9)$

$\therefore$ eqn of tangent plane: $\left(\underline{x} - (1,-1,-1)\right) \cdot (3,0,-9) = 0$

b) $X : \mathbb{R}^2 \to \mathbb{R}^3$ is clearly of class $C^1$. But,
$X_s \times X_t = (-3t^3, 0, 9s^2t^2) = 0$ at $(0,0,0)$ so
$X$ is not smooth.

§7.2 #1 $X(s,t) = (s, s+t, t) \quad 0 \le s \le 1,\ 0 \le t \le 2$

$X_s = (1,1,0) \quad X_t = (0,1,1) \quad X_s \times X_t = (1,-1,1)$

$\|X_s \times X_t\| = \sqrt{3}$

$$\iint_X (x^2+y^2+z^2)\,dS = \int_0^1\int_0^2 s^2+(s+t)^2+t^2\ \sqrt{3}\,ds\,dt$$

$$= \int_0^1\int_0^2 (2s^2+2t^2+2st)\sqrt{3}\,ds\,dt$$

$$= 2\sqrt{3}\int_0^1 \frac{s^3}{3}+st^2+s^2t\Big|_0^2\,dt$$

$$= 2\sqrt{3}\int_0^1 \frac{8}{3}+2t^2+4t\,dt$$

$$= 2\sqrt{3}\left(\frac{8}{3}+\frac{2}{3}+2\right)$$

$$= \frac{26\sqrt{3}}{3}$$

#2a) $X(s,t) = (s+t, s-t, st) \qquad (s,t) \in D$

$$X_s = (1,1,t) \quad X_t = (1,-1,s)$$

$$X_s \times X_t = (s+t, t-s, -2)$$

$$\|X_s \times X_t\| = \sqrt{(s+t)^2 + (t-s)^2 + (-2)^2}$$

$$= \sqrt{2s^2 + 2t^2 + 4}$$

$$\iint_X f\,dS = \iint 4\,dS = 4\iint_D \sqrt{2s^2+2t^2+4}\,dA$$

Convert to polar coordinates:

$$= 4\int_0^{\pi/2}\int_0^1 \sqrt{2r^2+4}\; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{2}{3}(2r^2+4)^{3/2}\Big|_0^1 d\theta$$

$$= \frac{\pi}{2}\cdot\frac{2}{3}\left(6^{3/2} - 4^{3/2}\right) = \frac{\pi}{3}\left(6^{3/2} - 8\right)$$

2b) $\displaystyle\iint_{X} F \cdot dS = \iint_{D} (F \circ X) \cdot (X_s \times X_t)\, dA$

$$= \iint_{D} (s+t, s-t, st) \cdot (s+t, t-s, -2)\, dA$$

$$= \iint_{D} (s+t)^2 - (s-t)^2 - 2st \; dA$$

$$= \iint 2st \, dA$$

$$= \int_0^{\pi/2}\int_0^1 2 r\cos\theta \, r\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 2r^3 \cos\theta \sin\theta \, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^1 u\,du = \frac{1}{4}$$

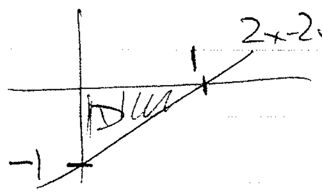
#3 $\iint \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\underline{n}\, dS$ ← upward pointing normal

$$= \iint (x,y,z)\cdot(2,-2,1)\, dS$$

$$= \iint_S 2x-2y+z\, dS$$

$X(u,v)=(u,v,2-2u+2v)$ $\quad X_u=(1,0,-2)$

$(u,v)\in D$ $\quad X_v=(0,1,2)$




$X_u\times X_v=(2,0,1)$

$\|X_u\times X_v\| = \sqrt{2^2+0^2+1^2} = \sqrt{5}$

$$\therefore \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D (2u-2v+2-2u+2v)\sqrt{5}\, du\, dv$$

$$= 2\sqrt{5} \times \text{Area of } D = \sqrt{5}$$

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Correction: We need $\underline{n}$ to be a unit normal, not just any normal. Mine was $\sqrt{5}$ long, so my answer was $\sqrt{5}$ to ~~b~~big. The answer is 1

4) The points in both parametrized surfaces satisfy:

$$3x^2 - 3y^2 = z^2$$

But this doesn't prove that the surfaces are the same. (I don't know why this was given as a hint).

4b $\quad F = (y, -x, z^2) \qquad X(s,t) = (s\cos t, s\sin t, 3s^2) \quad 0 \le s \le 2, \; 0 \le t \le 2\pi$

$$\iint_X F \cdot dS = \int_0^{2\pi}\int_0^2 F(X(s,t)) \cdot (X_s \times X_t)\, ds\, dt$$

$$= \int_0^{2\pi}\int_0^2 2 \times 6s^3 \cos t \sin t + 9s^5\, ds\, dt$$

$F(X(s,t)) = (s\sin t, -s\cos t, 9s^4)$

$$= \int_0^{2\pi} 3s^4 \sin t \cos t + \frac{3}{2}s^6 \Big|_0^2 dt$$

$X_s = (\cos t, \sin t, 6s)$

$X_t = (-s\sin t, s\cos t, 0)$

$X_s \times X_t = (6s^2\cos t, -6s^2 \sin t, s)$

$$= 3 \cdot 2^4 \int_0^{2\pi} \sin t \cos t + 2\, dt$$

$F(X(s,t)) \cdot X_s \times X_t = (6s^3 \cos t \sin t, 6s^3 \cos t \sin t, 6s^2)$

$$= 3 \cdot 2^6 \pi = 3 \cdot 64\pi = 192\pi$$

$0 \le s \le 1, \; 0 \le t \le 4\pi$

$Y(s,t) = (2s\cos t, 2s\sin t, 12s^2)$

$$\iint_Y F \cdot dS = \int_0^{4\pi}\int_0^1 (48s^2)^2 \times 2\cos t \sin t + 144 s^3\, ds\, dt$$

$Y_s = (2\cos t, 2\sin t, 24s)$

$Y_t = (-2s\sin t, 2s\cos t, 0)$

$$= \cancel{\int_0^{4\pi} 16s^4\, dt} \int_0^{4\pi} 36\, dt$$

$Y_s \times Y_t = (48s^2\cos t, 48s^2 \sin t, 4s)$

$$= 4\pi \cdot 36 = 144\pi$$

$F(Y(s,t)) = (48s^2 \sin t, -48s^2\cos t, 144 s^2)$

$F(Y(s,t)) \cdot Y_s \times Y_t = ((48s^2)^2 \sin t \cos t, (48s^2)^2 \sin t \cos t, 144 s^3)$

Y is not 1-1 so it is not a parametrization. X is.

#10 Compute $\iint z\,dS$ on top, bottom & lateral surfaces seperately :

Top $\displaystyle\iint_{Top} z\,dS = \iint_{Top} 4\,dS = 4 \times \text{Area of Top} = 4 \times \pi(3)^2$

$= 36\pi$

Bottom $\displaystyle\iint_{Bottom} z\,dS = \iint 0\,dS = 0$

lateral $X(s,t) = (3\cos s, 3\sin s, t)$ $\quad 0 \le s \le 2\pi$ ~~$0 \le 4 \le t$~~

$0 \le t \le 4$

$X_s = (-3\sin s, 3\cos s, 0)$

$X_t = (\ 0\ ,\ 0\ ,\ 1\ )$ outward orientation.

$X_s \times X_t = (3\cos s, 3\sin s, 0)$

$\|X_s \times X_t\| = 3$

$$\iint_{lateral} z\,dS = \int_0^4\int_0^{2\pi} t \times 3\, ds\,dt = 6\pi\int_0^4 t\,dt = 48\pi$$

$$\therefore \iint z\,dS = 36\pi + 0 + 48\pi = 84\pi$$

\#12 $\cancel{\iiint x}\ \iint xyz\,dS = \iint\limits_{Top} xyz\,dS + \iint\limits_{BOT} xyz\,dS + \iint\limits_{Lateral} xyz\,dS$

Note ~~zxy~~ integrand $\equiv 0$ on the bottom surface

so $\iint\limits_{bot} xyz\,dS = 0$.

Top $X(s,t) = (s\cos t,\ s\sin t,\ 4)$ $\quad 0 \le s \le 3$, $0 \le t \le 2\pi$

$$X_s = (\cos t, \sin t, 0)$$
$$X_t = (-s\sin t, s\cos t, 0)$$
$$X_s \times X_t = \cancel{s}\,(0, 0, s)$$
$$\|X_s \times X_t\| = s$$

$$\iint\limits_{Top} xyz\,dS = \int_0^{2\pi}\int_0^3 4s^3 \cos t \sin t\,ds\,dt$$

$$= \int_0^{2\pi} 3^4 \cos t \sin t\,dt$$

$$= 0$$

Lateral $\quad X(s,t) = (3\cos s, 3\sin s, t) \quad \begin{matrix} 0 \le s \le 2\pi \\ 0 \le t \le 4 \end{matrix}$

$\cancel{X_s \times X_t = (3\cos}$

$X_s = (-3\sin s, 3\cos s, 0)$

$X_t = (0, 0, 1)$

$X_s \times X_t = (3\cos s, 3\sin s, 0)$

$\|X_s \times X_t\| = 3$

$$\iint_{Lat} xyz\, dS = \int_0^4 \int_0^{2\pi} 3\cos s \cdot 3\sin s \cdot t \cdot 3\, ds\, dt$$

$$= \int_0^4 27t \underbrace{\left( \int_0^{2\pi} \cos s \sin s\, ds \right)}_{0} dt$$

$$= 0$$

$$\therefore \iiint xyz\, dS = 0 + 0 + 0 = 0$$