<u>Last time</u> • We have seen that there are two equivalent ways to orient regions in $\mathbb{R}^n$:

1. By an ordered n-tuple of vectors $(\vec{v}_1, \dots \vec{v}_n)$ [They should be linearly independent]
2. By a nowhere zero n-form $f(x_1 \dots x_n)\, dx_1 \wedge \dots \wedge dx_n$.

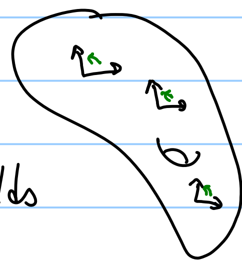
Similarly, an orientation of a curve C is either a continuous choice of a tangent vector or a nowhere zero 1-form.

An orientation of a surface $\Sigma \subset \mathbb{R}^3$ is either a

(a) a choice of a normal vector field

or (b) a choice of an ordered pair of tangent vector fields

or (c) a choice of a nowhere zero 2-form.



Now on to higher dimensions:

<u>Definition</u> A parameterized k-manifold in $\mathbb{R}^n$ is a differentiable map

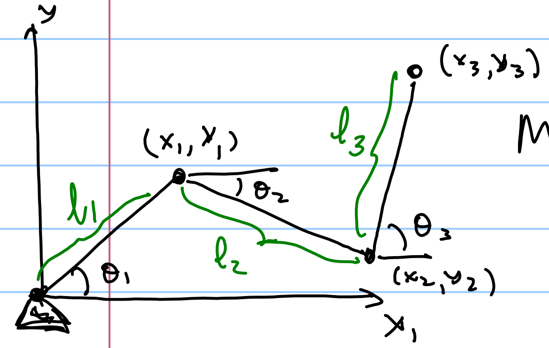
$$X : R \longrightarrow \mathbb{R}^n$$

where $R \subset \mathbb{R}^k$ is a region and X and its derivative DX are 1-1 except possibly on the boundary ∂R.

Note

- Any (parameterized) curve $\gamma : [a,b] \to \mathbb{R}^3$ is a parameterized 1-manifold.
- Any parameterized surface $\tau : R \to \mathbb{R}^3$ is a parameterized 2-manifold.

<u>Example</u> of a 3-manifold: space of configurations of a planar robot arm:



$$M = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 \;\middle|\; \begin{array}{l} x_1^2 + y_1^2 = l_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2 \\ (x_3 - x_2)^2 + (y_3 - y_2)^2 = l_3^2 \end{array} \right\}$$

l_1, l_2, l_3 <u>fixed</u> lengths

Parameterization?

$(x_1, y_1) = (l_1 \cos\theta_1, l_2 \sin\theta_1)$

$(x_2, y_2) = (x_1, y_1) + (l_2 \cos\theta_2, l_2 \sin\theta_2) = (l_1 \cos\theta_1 + l_2 \cos\theta_2, l_1 \sin\theta_1 + l_2 \sin\theta_2)$

$(x_3, y_3) = (x_2, y_2) + (l_3 \cos\theta_3, l_3 \sin\theta_3) = (l_1 \cos\theta_1 + l_2 \cos\theta_2 + l_3 \cos\theta_3, l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_3 \sin\theta_3)$

So $X : R = \{(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \mid 0 \le \theta_1 \le 2\pi, 0 \le \theta_2 \le 2\pi, 0 \le \theta_3 \le 2\pi\} \to \mathbb{R}^6$

$X(\theta_1, \theta_2, \theta_3) = (l_1 \cos\theta_1 + l_2 \cos\theta_2 + l_3 \cos\theta_3, l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_3 \sin\theta_3)$

We can integrate k-forms over oriented k-manifolds.
An orientation of a k-manifold $M \subset \mathbb{R}^n$ is determined by a nowhere-zero k-form.

Ex
$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1,\ x_3^2 + x_4^2 = 1\}$$
$\Omega = (x_1 dx_2 - x_2 dx_1) \wedge (x_3 dx_4 - x_4 dx_3)$ orients M.

A parameterization $X : R \to M$ is compatible with σ if $X^*\sigma$ gives R positive (standard) orientation.

Back to example

$X : \{(\theta_1, \theta_2) \mid 0 \le \theta_1 \le 2\pi,\ 0 \le \theta_2 \le 2\pi\} \to M$
$$X(\theta_1, \theta_2) = (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)$$

$$X^*\big((x_1 dx_2 - x_2 dx_1) \wedge (x_3 dx_4 - x_4 dx_3)\big) = (\cos\theta_1\, d\sin\theta_1 - \sin\theta_1\, d\cos\theta_1) \wedge (\cos\theta_2\, d\sin\theta_2 - \sin\theta_2\, d\cos\theta_2) = (\cos^2\theta_1 + \sin^2\theta_1)\, d\theta \wedge (\cos^2\theta_2 + \sin^2\theta_2)\, d\theta_2$$
$$= d\theta_1 \wedge d\theta_2.$$

$\therefore$ X is compatible with (the orientation defined by) Ω.

Suppose $X : R \to M$ is a parameterization of M compatible with its orientation, and τ is a k-form. Then
$$\int_M \tau := \int_R X^*\tau.$$

Back to example: $\tau = x_1 x_2\, dx_1 \wedge dx_2 + x_3 x_4\, dx_2 \wedge dx_4$.

$$\int_M \tau = \int_R X^*(x_1 x_3\, dx_2 \wedge dx_4) = \int_R \cos\theta_1 \cos\theta_2\, d(\sin\theta_1) \wedge d(\sin\theta_2) =$$
$$= \int_R \cos^2\theta_1 \cos^2\theta_2\, d\theta_1 \wedge d\theta_2 = \int_0^{2\pi}\!\!\int_0^{2\pi} \cos^2\theta_1 \cos^2\theta_2\, d\theta_1\, d\theta_2 = \left(\int_0^{2\pi} \tfrac{1}{2}(\cos 2\theta + 1)\, d\theta\right)^2 = \pi^2$$

Recall we are after (generalized)

Stokes' Theorem Let τ be a $(k-1)$ form on an oriented k-manifold M with boundary ∂M. Then

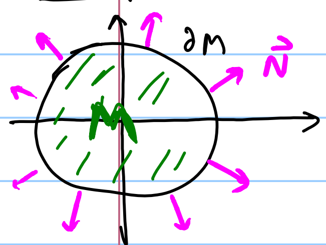
$$\int_M d\tau = \int_{\partial M} \tau.$$

Small problem To make sense of $\int_{\partial M} \tau$ we need to orient ∂M "correctly."

Here is how we do it: Choose an outward normal $\vec{N}$ at points of ∂M. Given an orientation form Ω on M define an orientation form $\Omega_{\partial M}$ on ∂M by

$$\boxed{\Omega_{\partial M} = \iota(\vec{N})\Omega}$$

Example $M = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$ $\quad \partial M = \{(x,y) \mid x^2+y^2 = 1\}$



$$\vec{N} = x\vec{i} + y\vec{j}$$

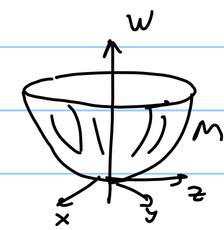
$$\Omega = dx \wedge dy$$

$$\iota(\vec{N})\Omega = \iota(x\vec{i}+y\vec{j})\, dx \wedge dy = dx(x\vec{i}+y\vec{j})\, dy - dy(x\vec{i}+y\vec{j})\, dx =$$

$$= x\,dy - y\,dx$$

We've seen last time $x\,dy - y\,dx$ orients the circle ∂M counterclockwise, with M on the left.

Example (cf example 3, p 500-502 of text).



$$M = \{(x,y,z,w) \in \mathbb{R}^4 \mid w = x^2+y^2+z^2, \quad x^2+y^2+z^2 \le 1\}$$

oriented by

$$\Omega = \iota(2x, 2y, 2z, -1)\, dx\wedge dy\wedge dz\wedge dw = 2x\, dy\wedge dz\wedge dw - 2y\, dx\wedge dz\wedge dw + 2z\, dx\wedge dy\wedge dw - (-1)\, dx\wedge dy\wedge dz.$$

Verify Stokes' thm for the 2-form $\omega = zw\, dx\wedge dy$

Solution

We parameterize M by $X: B \to \mathbb{R}^4$, $\quad B = \{(u_1,u_2,u_3) \in \mathbb{R}^3 \mid u_1^2+u_2^2+u_3^2 \le 1\}$

$$X(u_1,u_2,u_3) = (u_1, u_2, u_3, u_1^2+u_2^2+u_3^2).$$

$$X^*\Omega = 2u_1\, du_2\wedge du_3\wedge d(u_1^2+u_2^2+u_3^2) - 2u_2\, du_1\wedge du_3\wedge d(u_1^2+u_2^2+u_3^2) + 2u_3\, du_1\wedge du_2\wedge$$
$$\wedge\, d(u_1^2+u_2^2+u_3^2) + du_1\wedge du_2\wedge du_3 = 4u_1^2\, du_2\wedge du_3\wedge du_1 - 4u_2^2\, du_1\wedge du_3\wedge du_2 +$$
$$4u_3^2\, du_1\wedge du_2\wedge du_3 + du_1\wedge du_2\wedge du_3 = (4u_1^2+4u_2^2+4u_3^2+1)\, du_1\wedge du_2\wedge du_3$$

$\Rightarrow X$ is compatible with Ω.

Now $\int_M d\omega = \int_B X^*(d\omega) = \int_B X^*\left(d(zw\,dx\wedge dy)\right) = \int_B d\left(u_3\,(u_1^2+u_2^2+u_3^2)\,du_1\wedge du_2\right)$

$= \int_B \left((u_1^2+u_2^2+u_3^2) + u_3\cdot 2u_3\right) du_3\wedge du_1\wedge du_2$

$= \iiint_{u_1^2+u_2^2+u_3^2\le 1} (u_1^2+u_2^2+3u_3^2)\,du_1\,du_2\,du_3$. Now we change to spherical coordinates

and compute that the integral is $\frac{4\pi}{3}$.

Next $\partial M = \{(x,y,z,w) \mid x^2+y^2+z^2=1,\ w=1\}$, a 2-sphere.

Induced orientation: $\vec{n} = (x,y,z,2)$ points out of M.

$\Omega_{\partial M} = \iota(\vec{n})\Omega$ (computed at points of ∂M!) On ∂M, $w=1 \Rightarrow dw=0$.

$\Rightarrow \Omega$ on ∂M is $dx\wedge dy\wedge dz$.

$\Rightarrow \Omega_{\partial M} = \iota((x,y,z,2))\,dx\wedge dy\wedge dz = x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy$.

We parmeterize ∂M using spherical coordinates:

$Y: [0,\pi]\times[0,2\pi] \longrightarrow \partial M$, $Y(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi, 1)$

Long and tedious computation shows: $Y^*\left(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy\right)$

$= \sin\varphi\,d\varphi\wedge d\theta$. $\Rightarrow Y$ is compatible with $\Omega_{\partial M}$. Hence

$\int_{\partial M} \omega = \int_{[0,\pi]\times[0,2\pi]} Y^*\omega = \int_{[0,\pi]\times[0,2\pi]} \cos\varphi\cdot 1\ d(\sin\varphi\cos\theta)\wedge d(\sin\varphi\sin\theta) =$

$= \int_{[0,\pi]\times[0,2\pi]} \cos\varphi\,(\cos\theta\cos\varphi\,d\varphi - \sin\varphi\sin\theta\,d\theta)\wedge(\cos\varphi\sin\theta\,d\varphi + \sin\varphi\cos\theta\,d\theta) =$

$= \int_{[0,\pi]\times[0,2\pi]} \cos\varphi\,(\cos^2\theta\cos\varphi\sin\varphi + \sin\varphi\cos\varphi\sin^2\theta)\,d\varphi\wedge d\theta = \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi\sin\varphi\,d\varphi\,d\theta =$

$= \int_0^{2\pi}\left(-\frac{1}{3}\cos^3\varphi\Big|_0^{\pi}\right)d\theta = \left(\frac{1}{3} - \frac{(-1)^3}{3}\right)\cdot 2\pi = 4\pi/3$.