347 3/26/07

Note Title

Recall: a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of a sequence $\{a_n\}_{n=1}^{\infty}$ is a sequence is a composition of a strictly increasing map $k \mapsto n_k$ (from $\mathbb{N}$ to $\mathbb{N}$) and $n \to a_n$ (from $\mathbb{N}$ to $\mathbb{R}$).

That is, $\{a_{n_k}\}$ assignes to each $k \in \mathbb{N}$ the number a_{n_k} and $1 \leq n_1 < n_2 < n_3 < \ldots$

Recall also: Every convergent sequence is bounded. The converse is false: $a_n = (-1)^n$ is bounded but it doesn't converge.

Theorem Every bounded sequence has a convergent subsequence.

The proof use two ideas:

1) if an infinite set C is a union of two subsets A & B:

$$C = A \cup B,$$

then at least one, A or B, has to be infinite.

2) If a sequence $\{a_n\}$ has the property that

$$|a_{n+1} - a_n| \leq \frac{1}{2^n} \quad \text{for all } n,$$

then $\{a_n\}$ converges.

Why is (2) true? We argue that $\{a_n\}$ is Cauchy.

$$|a_{n+1} - a_n| \leq \frac{1}{2^n}$$

$$\begin{aligned}|a_{n+2} - a_n| &= |a_{n+2} - a_{n+1} + a_{n+1} - a_n| \\ &\leq |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n| \\ &\leq \frac{1}{2^{n+1}} + \frac{1}{2^n}\end{aligned}$$

$$|a_{n+3} - a_n| \leq |a_{n+3} - a_{n+2}| + |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|$$
$$\leq \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} + \frac{1}{2^n}$$
$$\vdots \quad (\text{ie. induction})$$

$$|a_{n+k} - a_n| \leq \frac{1}{2^{n+k-1}} + \frac{1}{2^{n+k-2}} + \cdots + \frac{1}{2^n} = \frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\right)$$
$$= \frac{1}{2^n} \frac{1 - \left(\frac{1}{2}\right)^k}{1 - 1/2} < \frac{1}{2^n} \cdot \frac{1}{1/2} = \frac{2}{2^n}$$

Since $\frac{2}{2^n} \to 0$, $\forall \varepsilon > 0$, $\exists N$ so that $n \geq N \Rightarrow \frac{2}{2^n} < \varepsilon$

Therefore, if $m \geq n \geq N$ then $|a_m - a_n| < \frac{2}{2^n} < \varepsilon$.
That is,
$$\{a_n\} \text{ is Cauchy}$$
$\square$

Now let's prove a special case of the Theorem: suppose $0 \leq a_n \leq 1$ for all n (the notation is simpler in this case).
We want to construct a convergent subsequence.
$[0,1] = [0, 1/2] \cup [1/2, 1]$.
The sequence has to land in one of the two subintervals infinitely many times. $\Rightarrow$ one of the two sets
$$\{n \in \mathbb{N} \mid a_n \in [0, 1/2]\}$$
$$\{n \in \mathbb{N} \mid a_n \in [1/2, 1]\}$$
is infinite.

Ex $a_n = \frac{1}{2}(1 + (-1)^n)$ $\quad \{n \in \mathbb{N} \mid a_n \in [0, 1/2]\} = \{1, 3, 5, 7, \ldots\}$
$\{n \in \mathbb{N} \mid a_n \in [1/2, 1]\} = \{2, 4, 6, \ldots\}$

Pick one that's infinite: $S_1 = \{ n \in \mathbb{N} \mid a_n \in [c_1, c_1 + 1/2]$

$c_1 = 0$ or $1/2$,

By well-ordering, S_1 has the smallest element. Call it n_1.

Then $a_{n_1} \in [c_1, c_1 + 1/2]$.

For example, if $a_n = \frac{1}{2}(1+(-1)^n)$ and $S_1 = \{2, 4, 6, \dots\}$, then $n_1 = 2$

$c_1 = 1/2$ and $a_{n_1} = 1$.

Next write

$$[c_1, c_1 + 1/4] \cup [c_1 + 1/4, c_1 + 1/2]$$

One of the two sets

$$\{ n \in \mathbb{N} \mid n > n_1,\ a_n \in [c_1, c_1 + 1/4] \}$$
$$\{ n \in \mathbb{N} \mid n > n_1,\ a_n \in (c_1 + 1/4, c_1 + 1/2] \}$$ is infinite

Pick one that's infinite:

$$S_2 = \{ n \in \mathbb{N} \mid n > n_1 \quad a_n \in [c_2, c_2 + 1/4] \}$$

$c_2 = c_1$ or $c_1 + 1/4$.

By well ordering principle, S_2 has the smallest element.

Call it n_2. Then $n_1 < n_2$, $a_{n_2} \in [c_2, c_2 + 1/4] \subseteq [c_1, c_1 + 1/2]$.

Hence $|a_{n_2} - a_{n_1}| \le \frac{1}{2}$.

(If $a_n = \frac{1}{2}(1+(-1)^n)$ & $S_1 = \{2, 4, 6 \dots\}$ then $S_2 = \{4, 6, 8, \dots\}$

$n_2 = 4$ and $a_{n_2} = 1$)

Keep going. We get:

a sequence of intervals

$$[c_1, c_1 + 1/2] \supset [c_2, c_2 + 1/4] \supset [c_3, c_3 + \tfrac{1}{8}] \supset \dots \supset [c_k, c_k + \tfrac{1}{2^k}] \supset \dots$$

and a subsequence $\{a_{n_k}\}$ with $a_{n_k} \in [c_k, c_k + \frac{1}{2^k}]$

Moreover $|a_{n_{k+1}} - a_{n_k}| \le \frac{1}{2^k}$ (since $a_{n_k}, a_{n_{k+1}} \in [c_k, c_k + \frac{1}{2^k}]$).

Therefore $\{a_{n_k}\}$ is Cauchy (see above).
Thus $\{a_{n_k}\}$ converges.
Note! since $0 \le a_{n_k} \le 1$, $0 \le \lim a_{n_k} \le 1$ as well.

Another example:

$$a_n = \begin{cases} \frac{1}{2^n} & n \text{ even} \\ 1 - \frac{1}{2^n} & n \text{ odd} \end{cases}$$

$a_1 = 1 - 1/2$
$a_2 = \frac{1}{2^2}$
$a_3 = 1 - 1/2^3$
$a_4 = \frac{1}{2^4}$
$\vdots$

What does the proof do?

$\{n \in \mathbb{N} \mid a_n \in [0, 1/2]\} = \{1\} \cup \{2, 4, 6, 8, \ldots\}$
$\{n \in \mathbb{N} \mid a_n \in [1/2, 1]\} = \{1, 3, 5, 7, \ldots\}$

If we pick

$S_1 = \{1\} \cup \{2, 4, 6, 8, 10, \ldots\}$

$n_1 = 1, \quad n_2 = 2, \quad n_3 = 4, \quad n_4 = 6, \ldots, n_k = 2(k-1), \ldots$

$\{a_{n_k}\} = \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^4}, \ldots, \frac{1}{2^{2(k-1)}} \ldots\}$

if we pick $S_1 = \{1, 3, 5, 7, \ldots$

$n_1 = 1, \quad n_2 = 3, \quad n_3 = 5, \quad n_4 = 7, \quad \ldots, n_k = 2k - 1, \ldots$

$a_{n_k} = 1 - \frac{1}{2^{2k-1}}$