MATH 347 summary

Note Title

- $\sqrt{2}$ is irrational
- quantifiers and negation
- triangle inequality; $||x|-|y|| \le |x-y| \quad \forall x,y$
- induction
- well-ordering principle; well-ordering $\Rightarrow$ induction
- divisibility of integers; division algorithm for integers; gcd; $d = ax+by$ for some $a,b \in \mathbb{Z} \Rightarrow \gcd(x,y) \mid d$.
- algorithm for writing $\gcd(x,y)$ as $ax+by$ for some $a,b \in \mathbb{Z}$.
- primes; any integer is a prime or a product of primes;
- $\gcd(a,c)=1$ and $c \mid ab \Rightarrow c \mid b$.
- if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$ (or both).
- unique factorization into primes.
- division algorithm for polynomials
- roots of polynomials; if α is a root of $p(x)$ then $x-\alpha \mid p(x)$.
- # of roots of $p(x) \le \deg p(x)$.
- gcd of two polynomials
- Ideals in $\mathbb{R}[x]$. Any ideal is of the form $\mathbb{R}[x]\cdot p(x)$ for some $p(x)$
- $\mathbb{C}$ is a field; $\bar{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$, $|z+w| \le |z|+|w|$ ...
- upper bound, sup, sup is unique, completeness of $\mathbb{R}$, Archimedian property of $\mathbb{R}$, lower bound, inf
- definition of convergence for a sequence; limits are unique
  sup as a limit
- monotone sequences, bounded sequences, bounded monotone sequences converge; convergent $\Rightarrow$ bounded.

- Limit theorems: if $a_n \to L$, $b_n \to M$, then $a_n + b_n \to L+M$, $a_n \cdot b_n \to L \cdot M$, $a_n / b_n \to L/M$ (provided $b_n \neq 0$, $M \neq 0$).
- $a_n \to 0$, $\{b_n\}$ bounded $\Rightarrow a_n b_n \to 0$
- if $\frac{|b_{n+1}|}{|b_n|} \to x$ and $0 \le x < 1$, then $b_n \to 0$.
- Series, convergent series, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $|q|<1$.
- Squeez theorem.
- Cauchy sequences. Cauchy $\Leftrightarrow$ convergent (in $\mathbb{R}$ and in $\mathbb{C}$)

  Subsequences; $a_n \to L \Rightarrow a_{n_k} \to L$ for any subsequence $\{a_{n_k}\}$ of $\{a_n\}$.
- Bolzano-Weirstrass theorem
- Comparison test: $0 \le b_n \le a_n$, $\sum a_n$ converges $\Rightarrow \sum b_n$ converges

  $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.
- $\sum a_n$ converges $\Rightarrow a_n \to 0$
- $\sum |z_n|$ converges $\Rightarrow \sum z_n$ converges
- Ratio test
- $\sum \frac{1}{n}$ diverges, $\sum \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$
- $e^{iz} = \cos z + i \sin z \quad \forall z \in \mathbb{C}$
- Continuity ($\varepsilon - \delta$ definition)
- $f: D \to \mathbb{R}$ is continuous at $L \in D \Leftrightarrow$ ($\forall$ sequence $\{x_n\}$ with $x_n \to L$, $f(x_n) \to f(L)$).
- If $f, g: D \to \mathbb{R}$ are continuous $\Rightarrow f+g$, $f \cdot g$ are continuous (f/g is also continuous provided $g(x) \neq 0 \quad \forall x \in D$.)
- $f(x) = x$ is continuous, polynomials are continuous
- If $f: [a,b] \to \mathbb{R}$ is continuous, then $\exists c \in [a,b]$ so that $f(c) \ge f(x)$
- for all $x \in [a,b]$.

- Intermediate value theorem.
- injective, surjective, bijective ; invertible, composition of functions
- finite, infinite, countable
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable, $\mathbb{R}$ is not countable.
- A, B countable $\Rightarrow A \times B$ is countable
- relations, equivalence relations, equivalence classes.
- $\mathbb{Z}/n$, modular arithmetic
- $\mathbb{R}[x]/p(x)$
- $\mathbb{Z}/n$, $\mathbb{R}[x]/p(x)$ are rings
- If p is a polynomial of degree n, then there is a bijection $\mathbb{R}^n \to \mathbb{R}[x]/p\mathbb{R}[x]$.
- $\mathbb{R}[x]/x^2+1$ is "the same" as $\mathbb{C}$.