Lecture # 35 Nov 02, 2005

Note Title 11/29/2005

Last time:

1) We defined the pull-back $F^*\omega$ of a k-form ω by a map $F: \mathbb{R}^k \to \mathbb{R}^n$.

Recall: it amounts to substitution.

For example,

$$(\cos t, \sin t)^* (x\,dy - y\,dx) =$$
$$\cos t \, d(\sin t) - \sin t \, d(\cos t).$$

2) We defined integration of a 2-form $\omega = f(x,y)\, dx \wedge dy$ over a region $D \subseteq \mathbb{R}^2$:

$$\int_D f(x,y)\, dx \wedge dy := \int_D f(x,y)\, dx\, dy.$$

Similarly, we define integration of a 1-form $f(t)\,dt$ over an interval $[a,b]$ by

$$\underset{\text{1-form}}{\int_{[a,b]} f(t)\,dt} := \underset{\text{function}}{\int_a^b f(t)\,dt}$$

Similarly we can define integration of a k-form on a region $D \subseteq \mathbb{R}^k$ for any k:

$$\int_D \cdots \int F(x_1 \ldots x_k)\, dx_1 \wedge \cdots \wedge dx_k =$$

$$\int_D \cdots \int F(x_1 \ldots x_k)\, dx_1 \cdots dx_k .$$

3) If we put pull-back and integration of k-forms together we get integration of

1-forms over curves:

$$\int_{\vec{x}} \alpha = \int_{[a,b]} (\vec{x})^* \alpha$$

where $\vec{x} : [a,b] \to \mathbb{R}^n$ is a parameterization.

2-forms over surfaces:

$$\iint_X \omega = \iint_D X^* \omega$$

where $X : D \to \Sigma$ is

a parameterization and $D \subseteq \mathbb{R}^2$ is a domain.

This can be generalized to arbitrary dimensions

Def. A parameterised <u>k-manifold</u> in $\mathbb{R}^n$ is a differentiable map
$$X : D \to \mathbb{R}^n$$
where $D \subseteq \mathbb{R}^k$ is a region and X is 1-1 (except possibly on the boundary ∂D).

<u>Thus:</u> A 0-manifold is a point
A 1-manifold is a curve
A 2-manifold is a surface...

We can integrate k-forms over k-manifolds:

$$\int \cdots \int_X \omega := \int \cdots \int_D X^* \omega$$

where ω is a k-form in $\mathbb{R}^n$ and $X : D \to \mathbb{R}^n$ a parameterized k-manifold.

Our goal: (generalized) Stokes's Theorem:

Thm Let M be a $k+1$ manifold with boundary ∂M and ω a k-form. Then

$$\int \cdots \int_{\partial M} \omega = \int \cdots \int_M d\omega.$$

Recall how d is defined:

If $f : \mathbb{R}^n \to \mathbb{R}$ is a function

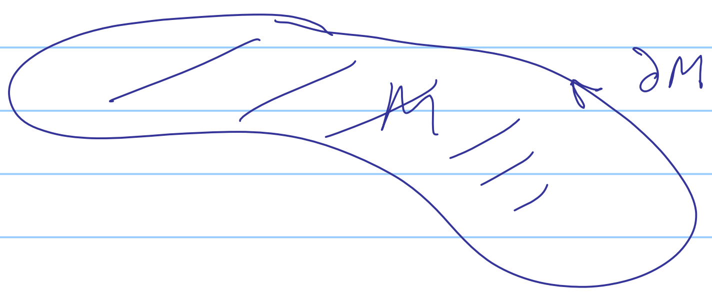
$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

If $\omega = \sum F_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a k-form, then

$$d\omega = \sum d(F_{i_1 \cdots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

"Example"

$M \subseteq \mathbb{R}^2$ a region.



$\alpha = P\,dx + Q\,dy$ a 1-form.

$$d\alpha = dP \wedge dx + dQ \wedge dy = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

$$\int_{\partial M} P\,dx + Q\,dy = \iint_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

This is Green's theorem.

"Example"

$\omega = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$, a form in $\mathbb{R}^3$ and $M \subseteq \mathbb{R}^3$ a region with boundary ∂M:

M

$$d\omega = dF_1 \wedge dy \wedge dz + dF_2 \wedge dz \wedge dx + dF_3 \wedge dx \wedge dy =$$

$$\left(\frac{\partial F_1}{\partial x} d \,,\, \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz\right) \wedge dy \wedge dz + \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz\right) \wedge dz \wedge dx + \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz\right) \wedge dx \wedge dy$$

$$= \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial F_3}{\partial z} dz \wedge dx \wedge dy = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz.$$

So we get

$$\iint_{\partial M} F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy = \iiint_M \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz$$

Gauss's theorem!